TRANSLATION

THE HYDRODYNAMICS OF AN EXPLOSION

By

Yu. S. Yakovlev

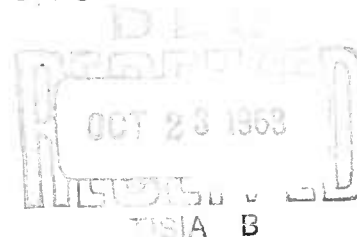# FOREIGN TECHNOLOGY DIVISION

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THE HYDRODYNAMICS OF AN EXPLOSION

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THE HYDRODYNAMICS OF AN EXPLOSION

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INTRODUCTION

The problems of applied theory of explosion attract the attention of a large circle of specialists.

Explosions are used for large-scale mining and earth-moving work: construction of channels, dams, roads, mine shafts, etc. Explosion is finding ever-increasing application in the oil industry (oil drilling), in processing of metals, and in other fields of the national economy. Seismic-explosion methods have found extensive use in the study of the structure of the earth's crust and in prospecting for useful minerals.

A study of the theory of explosion is also of great interest for shipbuilding engineers. One of the main requirements imposed upon a modern ship is a long life. This quality is attained by rational design of the hull structures, which makes provision, among the other theoretical cases, also for the action of dynamic loads. The theory of explosions is the basis for the determination of such loads.

The work by the Soviet school of hydrodynamics, and primarily by Kochin, Lavrent'yev, Sedov, Landau, Khristyanovich, Zel'dovich, Shimanskiy, Sadovskiy, Novozhilov, Stanyukovich, Pokrovskiy, Vlasov, Olisov, Patrashev, and other scientists, has laid the groundwork for the theory of explosion as an independent field of knowledge.

However, the original investigations by the foregoing authors are scattered among many publications. Even a cursory acquaintanceship with them calls for much time and labor.

There are practically no books in which the problem is developed sufficiently fully from a single unified point of view.

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This gap was filled to a considerable degree by the recently published monograph of F.A. Baum, K.P. Stanyukovich, and B.I. Shekhter, "The Physics of Explosions." It considers in detail questions of thermal chemistry, combustion, and brisance of explosives, the theory of detonation, and the theory of cumulation. Much attention is paid to problems in nonstationary gasdynamics. However, the problem of external forces and problems involving the action of an explosion on a structure are not investigated by the authors of "Physics of Explosions."

Yet it is precisely these problems that are of great interest to engineers, designers, and scientific workers.

This was precisely the motivation for writing the present book, which consists of four chapters.

In Chapter 1 — "General Laws Governing the Propagation of Shock Waves" — principal attention is paid to a consistent and sufficiently rigorous development of the principles of nonstationary gasdynamics as applied to nonsteady motions, the characteristic feature of which is the presence of discontinuity surfaces. The main material of the chapter is preceded by a brief exposition of thermodynamics.

Chapter 2 — "Explosion in an Unbounded Medium" — does not pretend to be complete in exposition, but allows judgment to be formed concerning methods of solving the problem. It also contains many formulas and relationships that are necessary for practical estimates. It was deemed appropriate to include in this chapter also the general laws of similarity theory, knowledge of which is essential to both the theoretician and particularly to the experimenter.

In Chapter 3 — "Simplest Boundary Problems of Explosion Theory" — are considered the reflection of an aerial shock wave from the earth's surface, of an underwater shock wave from the free surface and from the bottom of a water reservoir. An attempt was made to present, along

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with the results of the linear theory, some idea of the nonlinear effects, which have appreciable significance in the analysis of similar problems.

In the last section of the chapter are given semiempirical equations which permit an approximate estimate of the parameters of seismic-explosion waves near the earth's surface.

Finally, Chapter 4 — "Principal Aspects of the Problem of External Forces in the Case of Aerial and Underwater Explosions" — is devoted principally to problems of interaction between shock waves and a partition — one of the main problems of the theory of explosion and structural mechanics.

In the writing of this chapter, it was a particularly difficult problem to choose material so that without repeating the fundamental works in the field of strength under dynamic loads would nevertheless give a complete idea of the main methods for solving the problem.

The foregoing shows that the book does not, by far, treat all problems in the hydrodynamics of explosions. Thus, for example, we do not consider at all the theory of gas-bubble pulsation, surface phenomena in the case of underwater explosions, and other problems, the solution of which is usually presented under the assumption that the medium is incompressible.

This approach to the choice of material seemed to us justified, since it enables us to carry out the exposition by relying only on the main laws of gasdynamics. In addition, the theory of gas-bubble pulsation has been developed in great detail in the well-known book by R. Cowl "Underwater Explosions." For the same reason, little space has been allotted to problems of irregular and regular reflection of aerial shock waves from an absolutely rigid wall. Such problems are considered in detail in the monograph of R. Kurant and K. Fridrikh "Supersonic

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Flow and Shock Waves."

Tending to impart where possible an applied character to the exposition, the author deemed it advantageous to provide the main sections of the book with examples, which permit a deeper insight into the gist of the investigated phenomena and to acquire skill in the solution of practical problems.

The manuscript was evaluated during the course of preparation for publication by many specialists, whose remarks and preferences were taken into consideration to a great degree. The author expresses deep gratitude to those who participated in the discussion of the manuscript of the book.

Comments and desires with respect to the books should be addressed to: Leningrad, D-65, ul. Dzerzhinskogo 10, Sudpromgiz.

Chapter 1

GENERAL LAWS GOVERNING THE PROPAGATION OF SHOCK WAVES

§1. EXPLOSION AS VIEWED FROM THE HYDRODYNAMIC POINT OF VIEW

An explosion is defined as a chemical or nuclear reaction, as a result of which there is produced within a very short time interval within a definite volume, a high energy density resulting in the formation of a region of high pressures and temperatures. In the general case the distribution of the pressures and temperatures on the surface separating the explosion products from the surrounding medium is at the initial instant of time arbitrary, since the character of a reaction such as an explosion does not depend on the properties of the surrounding medium.

When the explosion energy propagates into the surrounding medium, surfaces are formed, on which the hydrodynamic elements of the fluid (pressure, density, temperature, velocity of motion of the particles) or else their time and distance derivatives change abruptly. Such surfaces will henceforth be called surfaces of strong and weak discontinuity, respectively.

If the pressure and normal component of the velocity vector of the fluid flow change abruptly on the surface of strong discontinuity, such a surface is called a nonstationary strong-discontinuity surface or a shock wave front.

If the pressure and normal component of the velocity on both sides of the discontinuity surface are the same, but the density and the temperature change abruptly, one speaks of a stationary strong-discon-

tinuity surface. A stationary strong-discontinuity surface separating the explosion products from the surrounding medium is frequently called the gas-bubble surface.

The main task of explosion theory is to study the unsteady motion of a fluid between two boundary surfaces — the front of the shock wave and the surface of the gas bubble. This motion of the fluid, as is well known, is determined by a system of partial differential equations. The analysis of such a system entails, generally speaking, unsurmountable mathematical difficulties. Consequently, from the very outset, on the basis of physical notions concerning the character of the explosion process, it is concluded that this system must be simplified by neglecting the volume forces and the viscosity forces. With such simplification explosion theory can be included as a specific section of nonstationary gasdynamics.

As was already mentioned earlier, the boundary conditions for the integration of the fundamental system of equations of gasdynamics are the conditions on the two discontinuity surfaces — the front of the shock wave and the surface of the gas bubble.

In view of the fact that the conditions on the surface of the gas bubble are assumed on the basis of the data of detonation theory, which is developed in sufficient detail in many fundamental papers, this problem is not treated here.

As regards the second boundary condition of the problem, it will become obvious that for a correct formulation of such a problem it is necessary to consider carefully first the general laws governing the propagation of shock waves, to which we now proceed.

Since the need for taking account of the compressibility of the medium calls for a thermodynamic treatment of the investigated phenomena, we shall first recall some of the thermodynamic concepts that are

necessary for what follows.

§2. BRIEF INFORMATION FROM THERMODYNAMICS

As is well known, there is a unique connection between the main parameters which determine the state of a substance, namely the mass density ρ, the pressure p, and the absolute temperature T.

An equation characterizing this relationship is called an equation of state of the given substance

$$F(p, \rho, T) = 0.$$

In place of the mass density one frequently uses the weight density $\gamma = g\rho$ kg/m^3 or its reciprocal, V m^3/kg, called the specific volume.

In kinetic theory of liquids and gases it is proved that the structure of the equation of state of any substance is determined by the relation

$$p = \frac{1}{V} f(V) T + \Phi(V). \tag{1.1}$$

Here the first term of (1.1) takes into account the motion and the second the interaction of the particles.

The form of the functions f and Φ is not established by the theory.

For an ideal gas, i.e., a gas with no interaction forces between the particles, we have $\Phi(V) = 0$ and $f(V) = $ const. The constant which replaces the function $f(V)$ in this case is called the gas constant and is designated R (m/deg).

In this case Eq. (1.1) reduces to the relation $pV = RT$, known as the Mendeleyev-Clapeyron equation.

In cases of practical importance, the air can be regarded as an ideal gas. Its equation of state is then the Mendeleyev-Clapeyron equation.

The equation of state of water has not yet been derived theoreti-

cally. Therefore, to solve practical problems one uses various empirical and semiempirical equations of state of water. We present two of them:

1) the Tait equation

$$V(T,p) = V(T,0)\left[1 - \frac{1}{n}\ln\left(1 + \frac{p}{B}\right)\right], \qquad (1.2)$$

where B is a function of the temperature only, usually specified in tabulated form;

2) an equation that approximates the experimental data of Bridgman for high pressures:

$$p = (109 - 93.7V)(T - 348) + 5010V^{-5.58} - 4310, \qquad (1.3)$$

where $\underline{p}$ is the pressure in atmospheres, V the specific volume in cm^3/g, and T the absolute temperature.

In thermodynamics one frequently uses the concept of internal energy of a substance. This is defined as the energy of motion and interaction of its structural particles (molecules and atoms).

The internal energy is determined completely by the parameters of state and consequently can be represented in one of the following forms:

$$\left.\begin{array}{l} u = u(p, T), \\ u = u(V, p), \\ u = u(T, V). \end{array}\right\} \qquad (1.4)$$

Being a function of the main parameters, the value of the internal energy depends only on the initial and final states of the substance and does not depend on the character of the dynamic processes that have led to the change in state.

Mathematically this fact is characterized by the equation

$$\oint du = 0. \qquad (1.5)$$

The parameters of state are related in basically different fashion with the heat acquired or lost by the substance. The character of the process plays in this case an important role.

Accordingly, the value of the specific heat $c = dQ/dT$ also changes.

We shall henceforth deal essentially with the specific heat of a process occurring at a constant volume, usually denoted c_v, and the specific heat of a process occurring at constant pressure, c_p.

The study of thermodynamic phenomena is based on two main laws. The first law of thermodynamics is the consequence of the law of energy conservation.

It is usually written in the form

$$dQ = du + A \sum dW \qquad (1.6)$$

and expresses the simple fact that the heat absorbed by a substance is consumed in changing its internal energy and performing work of varying type. Since the work is usually expressed in mechanical units, the second term of (1.6) is preceded by a coefficient called the thermal equivalent of work

$$A = 1/427 \text{ kcal/kg-m.}$$

In thermodynamics it is, for the most part, necessary to consider homogeneous systems,* in which the only form of work is work of expansion. Equation (1.6) then assumes the form

$$dQ = du + Ap dV. \qquad (1.7)$$

It is easy to see that in the case of an isochoric process, i.e., a process occurring at constant volume, all the heat supplied goes to increase the internal energy of the substance.

We shall consider the internal energy as a function of the temperature and specific volume, and write for it an expression for the total differential

$$du = \left(\frac{\partial u}{\partial T}\right)_v dT + \left(\frac{\partial u}{\partial V}\right)_T dV. \qquad (1.8)$$

Substituting (1.8) into (1.7), we obtain

$$dQ = \left(\frac{\partial u}{\partial T}\right)_v dT + \left[\left(\frac{\partial u}{\partial V}\right)_T + Ap\right] dV. \qquad (1.9)$$

For an isochoric process $dV = 0$ and Eq. (1.9) assumes the form $dQ = (\partial u/\partial T)_V dT$, hence

$$c_v = \left(\frac{dQ}{dT}\right)_v = \left(\frac{\partial u}{\partial T}\right)_v. \tag{1.10}$$

It can be shown that for an ideal gas the internal energy is a function of the temperature only, and the value of the specific heat c_V is constant. Consequently,

$$du = c_v dT, $$
$$dQ = c_v dT + Ap\, dV; \tag{1.11}$$
$$u = c_v T. \tag{1.12}$$

Proceeding to consideration of an isobaric process, i.e., a process occurring at constant pressure, we obtain by differentiating the Mendeleyev-Clapeyron equation $pV = RT$ with $p = const$

$$pdV = RdT. \tag{1.13}$$

From (1.11) we have

$$dQ_p = c_v dT + AR\, dT.$$

Since

$$\frac{dQ_p}{dT} = c_p, \tag{1.14}$$

the connection between the specific heat at constant pressure and at constant volume for an ideal gas is obtained in the form

$$c_p = c_v + AR. \tag{1.15}$$

Widely used in thermodynamics is the entropy function S (cal/kg-deg), defined by the relation

$$dS = dQ/T. \tag{1.16}$$

It is possible to separate from the total amount of heat that part which is the result of the work of friction forces (Q_{vnutr}). It is obvious that this part of the heat is always essentially positive. The remaining part of the heat (Q_{vneshn}) can be either positive or neg- ative (the heat can be either delivered to or removed from the sub-

stance). In accordance with the foregoing we have

$$dS = \frac{dQ_{внешн} + dQ_{внутр}}{T}. \qquad (1.17)$$

A process in which there is no heat transfer between the external and internal media is called adiabatic.

It follows from (1.17) that in the general case of an adiabatic process $dS \neq 0$. A process in which the entropy does not change ($dS = 0$) is possible in a nonviscous medium and is called isentropic, or an ideal adiabatic process.

Since

$$dQ_{внутр} \geqslant 0,$$

we have

$$dS \geqslant \frac{dQ_{внешн}}{T}. \qquad (1.18)$$

For an isolated system $dQ_{vneshn} = 0$ and Expression (1.18) assumes the form $dS \geq 0$.

Consequently, the entropy of an isolated system can only increase, and only for an ideal system can it remain constant. This is the content of the second law of thermodynamics.

In statistical thermodynamics it is proved that the growth of entropy of an isolated system is connected with the transition of the system from a less probable state into a more probable one.

Let us find an expression for the value of the entropy of an ideal gas in terms of the parameters of state.

We previously had

$$dQ = c_v \, dT + Ap \, dV. \qquad (1.11)$$

Consequently,

$$dS = \frac{dQ}{T} = c_v \frac{dT}{T} + Ap \frac{dV}{T}.$$

Since for T = pV/R, we have

we obtain

$$AR = c_p - c_v;$$
$$k = \frac{c_p}{c_v},$$

$$dS = c_v \frac{dT}{T} + c_v(k-1)\frac{dV}{V}.$$

Integrating this equation we obtain

$$S - S_0 = c_v \ln T + c_v(k-1)\ln V$$

or

$$\left.\begin{aligned} TV^{k-1} &= e^{\frac{S-S_0}{c_v}} \text{ const,} \\ pV^k &= e^{\frac{S-S_0}{c_v}} \text{ const,} \\ \frac{p}{\rho^k} &= e^{\frac{S-S_0}{c_v}} \text{ const.} \end{aligned}\right\} \qquad (1.19)$$

It is obvious that an isentropic process $(S - S_0 = 0)$ will be characterized by the satisfaction of one of the following relations:

$$\left.\begin{aligned} TV^{k-1} &= \text{const,} \\ pV^k &= \text{const,} \\ \frac{p}{\rho^k} &= \text{const.} \end{aligned}\right\} \qquad (1.20)$$

These relations are known as the Poisson adiabatic curves.
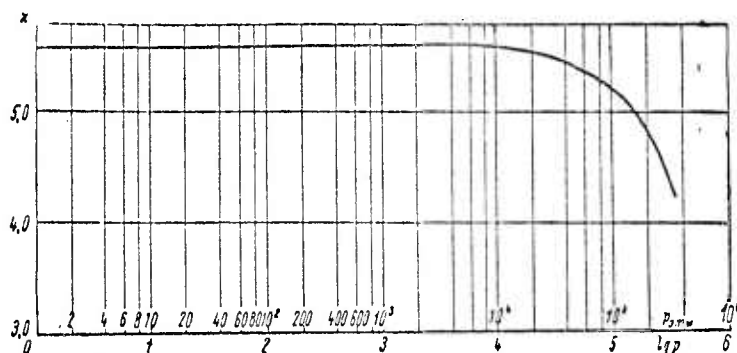
Fig. 1. Variation of the adiabatic exponent as a function of the pressure.

The value of the adiabatic exponent k depends on the number of atoms in the gas molecule. For diatomic gases, and particularly for air, we have $k = 1.4$.

Using the equation of state of water (1.3) we can, after going

through derivations that are in principle analogous to those given above, establish the following adiabatic condition for water:

$$\frac{p+C}{p^*} = \left(\frac{\rho}{\rho^*}\right)^{\kappa(S)},$$

(1.21)

where C, ρ^*, and p* are constants with values

$$C = 5400 \text{ kg/cm}^2; \quad \rho^* = 2.53 \text{ g/cm}^3; \quad p^* = 912,000 \text{ kg/cm}^2.$$

The coefficient κ depends on the entropy of the system and ranges from 5.55 to 4.60 in the range of variation of the initial values $1 \leq \leq p \leq 250 \cdot 10^3$ atm and $T < 2000°K$.

It is characteristic that this coefficient varies slowly at the beginning (Fig. 1).

Thus, $\kappa = 5.55$ for $p = 1$ atm and $\kappa = 5.45$ for $p = 30,000$ atm, i.e., the variation of the coefficient κ over this interval of pressure is merely 2%.

This fact indicates that in underwater-explosion conditions the propagation of shock waves with pressures up to 30,000 atm on the front can be considered in the assumption that the process is isentropic.

The isentropic condition for $p < 30 \cdot 10^3$ atm can be obtained from the equation of state in Tait's form, and has the form

$$\frac{p+B}{\rho^n} = \frac{p_0 + B}{\rho_0^n},$$

(1.22)

where B and $\underline{n}$ are constants with values $B = 3045$ kg/cm^2 and $n = 7.15$.

It is easy to note that Eqs. (1.21) and (1.22) are identical in form and are quite close to the isentropic condition of an ideal gas.

This circumstance makes it possible in many cases to extend the solutions obtained for an ideal gas to include the case of water.

It must be borne in mind, however, that the exponents $\underline{n}$ and κ actually imply a different sense.

§3. STRONG EXPLOSIONS IN GASDYNAMICS

Upon propagation of the explosion energy, as indicated above,

strong discontinuity surfaces are formed. It is established that the general relations which take place on such surfaces. It is obvious that for this purpose we cannot use differential equations that assume continuous variation of the functions. Consequently, a study of discontinuity surfaces is best started with an analysis of the equations of gasdynamics in integral form.

As is well known, the equations of gasdynamics are based on three general laws of physics: the law of conservation of matter, the law of conservation of momentum, and the law of conservation of energy.

Let us consider some elementary liquid volume τ, bounded by a surface S, for two instants of time t_1 and t_2 which are infinitesimally close to each other. On the basis of the law of mass conservation we can write

$$\left(\iiint_\tau \rho \, d\tau \right)_{t=t_2} = \left(\iiint_\tau \rho \, d\tau \right)_{t=t_1}, \qquad (1.23)$$

where ρ stands for the mass density of the liquid.

Equation (1.23) is usually called the continuity equation.

According to the law of momentum conservation, the increment in the momentum is equal to the impulse of the acting forces, and therefore in the considered case of motion of a nonviscous liquid we have

$$\left(\iiint_\tau \rho \vec{v} \, d\tau \right)_{t=t_2} - \left(\iiint_\tau \rho \vec{v} \, d\tau \right)_{t=t_1} = -\int_{t_1}^{t_2} \left(\iint_S p \vec{n} \, dS \right) dt, \qquad (1.24)$$

where $\underline{v}$ is the velocity vector of liquid motion, $\underline{p}$ is the pressure, and $\underline{n}$ is a unit upward-normal vector.

The minus sign in the right half of Eq. (1.24) is the consequence of the well-known property of an ideal fluid, whereby an ideal fluid can experience only normal stresses, which are compressive stresses.

Equation (1.24) is called the equation of motion.

Finally, the law of energy conservation, according to which the

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change in the kinetic energy of an isolated liquid volume, added to
the increment of its internal energy, is equal to the work of the ex-
ternal forces applied to this volume, can be written in the form

$$\left(\iiint_\tau \rho\,\frac{\vec{v}\,\vec{v}}{2}\,d\tau\right)_{t=t_1} - \left(\iiint_\tau \rho\,\frac{\vec{v}\,\vec{v}}{2}\,d\tau\right)_{t=t_1} +$$

$$+ \frac{1}{A}\left(\iiint_\tau \rho\,u\,d\tau\right)_{t=t_1} - \frac{1}{A}\left(\iiint_\tau \rho\,u\,d\tau\right)_{t=t_1} =$$

$$= -\int_{t_1}^{t_1}\left(\iint_S p\vec{v}\,\vec{n}\,dS\right)dt. \tag{1.25}$$

Here <u>u</u> denotes in addition the internal energy, and A is the
thermal equivalent of work, $A = 1/427$ kcal/kg-m.

Relation (1.25) is customarily called the energy equation.

All three equations (1.23), (1.24), and (1.25) can be written in
the following general form

$$\left(\iiint_\tau a\,d\tau\right)_{t=t_1} - \left(\iiint_\tau a\,d\tau\right)_{t=t_1} = \int_{t_1}^{t_1}\left(\iint_S c_n\,ds\right)dt, \tag{1.26}$$

where in the first case

$$a = \rho, \quad c_n = 0; \tag{1.27}$$

in the second

$$a = \rho\vec{v}, \quad c_n = -p\vec{n}; \tag{1.28}$$

and in the third

$$a = \rho\,\frac{\vec{v}\,\vec{v}}{2} + \rho\,\frac{u}{A}, \quad c_n = -p\vec{v}\,\vec{n}. \tag{1.29}$$

Let us assume that there exists a single surface Σ passing through
the points of the volume τ and moving in space, passage through which
causes the functions <u>a</u> and c_n to experience a discontinuity. Let the
equation of this surface be

$$F(x, y, z, t) = 0. \tag{1.30}$$

The surface Σ divides the space into two regions. On one side of
the surface we have F(x, y, z, t) < 0, and on the other F(x, y, z, t) >
> 0.
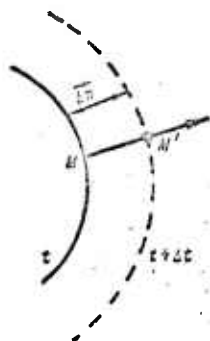
Fig. 2. Illustrating the definition of the rate of displacement of the discontinuity surface.

We shall call the first region negative and agree to denote the values to which some function b(x, y, z, t) tends on approaching Σ, remaining in the negative region by b_; the second region, will be called positive, and the values of b corresponding to it will be called b_+.

The difference b_+ − b_ will be denoted by [b]

$$b_+ - b_- = [b]$$

and will called the discontinuity or jump of the function b on the surface Σ. The surface Σ itself will then be called the discontinuity surface of the function b.

We introduce the concept of the rate of displacement of the surface Σ at a given point, the value of which is defined by the relation (Fig. 2)

$$N = \lim_{\Delta t \to 0} \frac{\Delta n}{\Delta t}. \tag{1.31}$$

Let us calculate the rate of displacement N. Since the equation of the surface Σ at the instant of time t + Δt is

$$F(x + \Delta x, y + \Delta y, z + \Delta z, t + \Delta t) = 0,$$

we obtain, by expanding this function in a Taylor series

$$F(x + \Delta x, y + \Delta y, z + \Delta z, t + \Delta t) = F(x, y, z, t) + \frac{\partial F}{\partial x} \Delta x + \\ + \frac{\partial F}{\partial y} \Delta y + \frac{\partial F}{\partial z} \Delta z + \frac{\partial F}{\partial t} \Delta t + \text{б. м. в. п.}^*, \tag{1.32}$$

but

$$\Delta x = \Delta n \cos(n, x) = \Delta n \frac{\frac{\partial F}{\partial x}}{\sqrt{\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 + \left(\frac{\partial F}{\partial z}\right)^2}},$$

$$\Delta y = \Delta n \cos(n, y) = \Delta n \frac{\frac{\partial F}{\partial y}}{\delta},$$

$$\Delta z = \Delta n \cos(n, z) = \Delta n \frac{\frac{\partial F}{\partial z}}{\delta}, \tag{1.33}$$

where

$$\delta = \sqrt{\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 + \left(\frac{\partial F}{\partial z}\right)^2}.$$

Substituting (1.33) into (1.32), we obtain

$$0 = 0 + \Delta n \frac{\left(\frac{\partial F}{\partial x}\right)^2}{\delta} + \Delta n \frac{\left(\frac{\partial F}{\partial y}\right)^2}{\delta} + \Delta n \frac{\left(\frac{\partial F}{\partial z}\right)^2}{\delta} + \frac{\partial F}{\partial t} \Delta t + \text{б. м. в. п.},$$

or

$$-\frac{\partial F}{\partial t} = \frac{\Delta n}{\Delta t} \delta + \text{б. м. в. п.}$$

Going to the limit as $\Delta t \to 0$, we get

$$\lim_{\Delta t \to 0} \frac{\Delta n}{\Delta t} = N = -\frac{\partial F}{\partial t} \frac{1}{\delta} = -\frac{\frac{\partial F}{\partial t}}{\sqrt{\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 + \left(\frac{\partial F}{\partial z}\right)^2}}. \qquad (1.34)$$

The quantity N is called the rate of displacement of the discontinuity surface. This rate determines the motion of the discontinuity surface relative to a stationary observer.

It is of interest to consider also another quantity, the rate of displacement of the discontinuity surface relative to a liquid moving ahead or behind the discontinuity front. It is called the rate of propagation of the discontinuity surface and is denoted by θ, where it is obvious that

$$\theta = N - v_n, \qquad (1.35)$$

where v_n is the value of the normal component of the particle velocity.

Since

$$v_n = v_x \cos(n, x) + v_y \cos(n, y) + v_z \cos(n, z),$$

$$v_x = \frac{dx}{dt}, \quad v_y = \frac{dy}{dt}, \quad v_z = \frac{dz}{dt},$$

and

we have

$$0 = -\frac{\partial F}{\partial t} \frac{1}{\delta} - \frac{\partial F}{\partial x} \frac{dx}{dt} \frac{1}{\delta} - \frac{\partial F}{\partial y} \frac{dy}{dt} \frac{1}{\delta} - \frac{\partial F}{\partial z} \frac{dz}{dt} \frac{1}{\delta} =$$

$$= -\frac{\frac{\partial F}{dt}}{\sqrt{\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 + \left(\frac{\partial F}{\partial z}\right)^2}}. \qquad (1.36)$$

Using the concept of rate of propagation of the discontinuity sur-face, let us establish the relations that connect (via the hydrody-namic equation) the jumps (discontinuity) of different hydrodynamic elements.
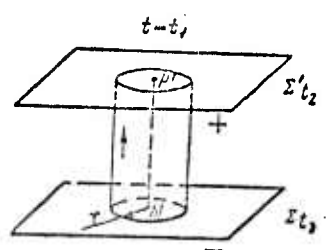
Fig. 3. Position of the discontinuity surfaces at the in-stant of time t_1.
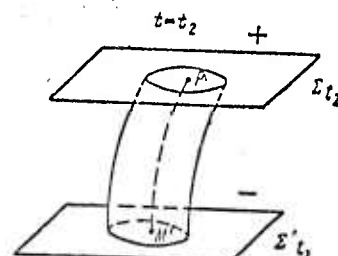
Fig. 4. Position of discontinuity sur-faces at the instant of time t_2.

Let us consider the surface Σ at two infinitesimally close in-stants of time t_1 and t_2.

We denote by Σ_{t_1} the position of the discontinuity surface at the instant t_1 and by Σ'_{t_2} the position at the instant t_1 of those points of the liquid, which are at the instant t_2 on the discontinuity sur-face (Fig. 3).

On the surface Σ_{t_1} we single out a point M and construct a small cylinder of radius $\underline{r}$ with axes coinciding with the normal to Σ_{t_1}. The height of the small cylinder will be the distance between the surfaces Σ_{t_1} and Σ'_{t_2}. We assume that $\Delta t/r$ is an infinitesimally small quantity. The volume of the cylinder is taken to be the integration volume τ up to the instant t_1.

At the instant t_2, owing to the motion of the liquid, the point M moves over to the point M'. The surface Σ_{t_1} occupies, after displace-ment and deformation, a certain position Σ'_{t_1}; finally, the points of the surface Σ'_{t_2} go over into the points of the discontinuity surface Σ_{t_2}, corresponding to the instant t_2 (Fig. 4). The small cylinder be-

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comes deformed, and whereas at the instant t_1 it is entirely in the positive region, by the instant t_2 it is located in the negative region corresponding to this instant of time.

Turning first to the left half of (1.26), let us estimate the values of the integration volumes $(\tau)_{t_1}$ and $(\tau)_{t_2}$; for the first we ·have*

$$\pi r^2 0_+ (t_2 - t_1) + e_1 r^2 (t_2 - t_1),$$

and for the second

$$\pi r^2 0_- (t_2 - t_1) + e_2 r^2 (t_2 - t_1).$$

In both cases we take θ to mean here the values of the rate of propagation of the discontinuity surface at one and the same point M and at one and the same instant of time t_1, but calculated for an approach to the surface Σ_{t_1} from opposite sides.

Thus,

$$\left(\iiint_{(\tau)} a \, dt \right)_{t_1} = a_+ \pi r^2 0_+ (t_2 - t_1) + e_3 r^2 (t_2 - t_1);$$

$$\left(\iiint_{(\tau)} a \, dt \right)_{t_2} = a_- \pi r^2 0_- (t_2 - t_1) + e_4 r^2 (t_2 - t_1)$$

and, consequently,

$$\left(\iiint_\tau a \, dt \right)_{t_1} - \left(\iiint_\tau a \, dt \right)_{t_2} = -\pi r^3 (t_2 - t_1)(a_+ 0_+ - a_- 0_-) +$$
$$+ e_5 r^2 (t_2 - t_1) = -\pi r^2 (t_2 - t_1) [a0] + e_5 r^2 (t_2 - t_1).$$

Let us proceed now to the right half of (1.26) (Fig. 5). On the basis of analogous reasoning we obtain

$$\int_{t_1}^{t_2} \left(\int_S c_n \, dS \right) dt = \int_{S_{cp}} c_n \, dS \Delta t = \int_{S_{бок}} c_n \, dS \Delta t + \int_{S_1} c_n \, dS \Delta t +$$
$$+ \int_{S_2} c_n \, dS \Delta t = 0 (1) r (\Delta t)^2 - c_{n-} \pi r^2 \Delta t + c_{n+} \pi r^2 \Delta t + e_6 \pi r^2 \Delta t.**$$

Gathering together the obtained estimates, we get

$$\left(\iiint_\tau a \, d\tau \right)_{t=t_2} - \left(\iiint_\tau a \, d\tau \right)_{t=t_1} - \int_{t_1}^{t_2} \left(\iint_S c_n \, dS \right) dt =$$
$$= -([a0] + [c_n]) \pi r^2 \Delta t + 0 (1) r \Delta t^2 + e_7 r^2 \Delta t.$$

Going to the limit and letting first $\Delta t = t_2 - t_1$ and then also $r \rightarrow 0$, we obtain ultimately

$$[a\theta] + [c_n] = 0 \tag{1.37}$$

or, after substituting in (1.37) the values of $\underline{a}$ and c_n from (1.27)-(1.29):

$$\left.\begin{array}{c} [\rho\theta] = 0, \\ [\rho\vec{v}\theta] = [p]\,\vec{n}, \\ \left[\rho\theta\,\dfrac{v^2}{2} + \dfrac{p}{A}\,\theta u\right] = [pv_n]. \end{array}\right\} \tag{1.38}$$

Inasmuch as the product $\rho\theta$ is not discontinuous, it can be taken outside the discontinuity sign.

As a result we obtain

$$[\rho\theta] = 0; \tag{1.39}$$

$$\rho\theta\,[\vec{v}] = [p]\,\vec{n}; \tag{1.40}$$

$$\rho\theta\left[\frac{\vec{v}\cdot\vec{v}}{2} + \frac{1}{A}u\right] = [pv_n]. \tag{1.41}$$
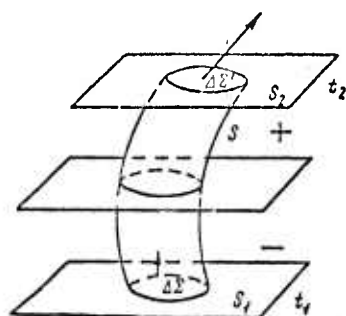
Fig. 5. Illustrating the evaluation of the integral over the lateral surface of the elementary cylinder.

Relations (1.39)-(1.41) between the quantities θ, ρ, $\underline{v}$, $\underline{p}$, and $\underline{u}$ on the two sides of the discontinuity surface are called the conditions of dynamic compatibility.

As follows from the foregoing exposition, the conditions of dynamic compatibility are a mathematical formulation of the general laws of conservation of mass, momentum, and energy on the discontinuity surfaces.

If $\theta = 0$, then the discontinuity surface is stationary.* On such a surface $[p] = 0$ and $[v_n] = 0$, but to the contrary $[\rho]$ has an arbitrary value. Examples of a stationary discontinuity surface can be: the wavy surface of a river, the surface of a hot or cold front (in meteorology) and, as already mentioned, the surface of the gas bubble in an underwater explosion.

If $\theta \neq 0$, the discontinuity surface is called nonstationary.

On such a surface $[p] \neq 0$; $[v_n] \neq 0$.*

The jump in the tangential component of velocity is in this case, to the contrary, equal to zero, as can be readily verified by taking the scalar products of both halves of (1.40) with a unit vector perpendicular to $\vec{n}$.

As was already mentioned earlier, an example of a nonstationary discontinuity surface may be the front of a shock wave.

We have considered cases when the hydrodynamic elements themselves are discontinuous on the surface.

Such surfaces are called strong-discontinuity surfaces. If a function is continuous on going through the surface but its derivative of any order is discontinuous, such a surface is called a weak-discontinuity surface.

In general texts on gasdynamics it is proved that the weak-discontinuity surface always propagates with the local velocity of sound. We recall that the local velocity of sound is determined by the equation $a = \sqrt{dp/d\rho}$.

§4. THE CONDITIONS OF DYNAMIC COMPATIBILITY. THE DYNAMIC ADIABATIC CURVE

The conditions of dynamic compatibility, established in the preceding section, enable us to draw several important conclusions concerning the propagation of shock waves in a liquid and to obtain some equations that are convenient for computations.

We note first of all that the three equations (1.39)-(1.41) contain seven variables: N, ρ_+, ρ_-, $\vec{v}_+$, $\vec{v}_-$, p_+, p_-.** Therefore if we know the hydrodynamic parameters of the unperturbed medium (ρ_+, v_+, p_+), it is sufficient to specify any one element on the discontinuity surface in order uniquely to determine all the remaining elements. Let us calculate the rate of propagation θ of the surface of strong discontinuity.

We take the scalar products of both halves of (1.40) with $\vec{n}$. We obtain $\rho\theta[v_n] = [p]$, but since $\theta = N - v_n$, we have $[v_n] = -[\theta]$.

Consequently, $-\rho\theta[\theta] = [p]$ or, what is the same, $-\rho\theta[\rho\theta/\rho] = [p]$.

Since the product $\rho\theta$ is not discontinuous, we have $-\rho^2\theta^2(1/\rho_+ - 1/\rho_-) = [p]$, or

$$\frac{\rho^2\theta^2}{\rho_+\rho_-} = \frac{[p]}{[\rho]}.$$

From the last relation it follows that

$$\theta_-^2 = \frac{\rho_+}{\rho_-}\frac{[p]}{[\rho]}; \tag{1.42}$$

$$\theta_+^2 = \frac{\rho_-}{\rho_+}\frac{[p]}{[\rho]}. \tag{1.43}$$

Return to the conditions of dynamic compatibility (1.39)-(1.41), and transform the last of these equations:

$$\rho\theta\left[\frac{\vec{v}\cdot\vec{v}}{2}\right] + \frac{\rho\theta}{A}[u] = [pv_n].$$

We first eliminate the quantity $\underline{v}$ from the written relation. For this purpose we use the second of the dynamic-compatibility conditions:

$$\rho\theta[\vec{v}] = [p]\vec{n}. \tag{1.40}$$

Taking the scalar product of both halves of (1.40) with $(\vec{v}_+ + \vec{v}_-)$, we obtain

$$\rho\theta(\vec{v}_+ - \vec{v}_-)(\vec{v}_+ + \vec{v}_-) = [p](\vec{v}_{n+} + \vec{v}_{n-});$$
$$\rho\theta[v^2] = [p](v_{n+} + v_{n-}).$$

Consequently,

$$[p](v_{n+} + v_{n-}) + 2\frac{\rho\theta}{A}[u] - 2(p_+v_{n+} - p_-v_{n-}) = 0;$$

$$p_+v_{n+} - p_-v_{n+} + p_+v_{n-} - p_-v_{n-} - 2p_+v_{n+} +$$
$$+ 2p_-v_{n-} + \frac{2\rho\theta}{A}[u] = 0.$$

$$-p_+v_{n+} + p_+v_{n-} - p_-v_{n+} + p_-v_{n-} + \frac{2\rho\theta}{A}[u] = 0;$$

$$p_+(v_{n-} - v_{n+}) + p_-(v_{n-} - v_{n+}) + \frac{2\rho\theta}{A}[u] = 0;$$

$$-(p_+ + p_-)[v_n] + \frac{2\rho\theta}{A}[u] = 0;$$

but

$$[v_n] = - [\theta] = - \left[\frac{p\theta}{\rho}\right] = - p\theta \left[\frac{1}{\rho}\right].$$

Thus

$$u_- - u_+ = \frac{A}{2} (p_+ + p_-) \left[\frac{1}{\rho}\right]. \qquad (1.44)$$

Equation (1.44) establishes the connection between the parameters of the medium on the surface of a nonstationary strong discontinuity and is called the equation of the dynamic or shock adiabatic curve.

In the case when the investigation concerns an ideal gas, the value of the internal energy is determined by the product

$$u = c_v T,$$
$$p = \rho RT$$

and

$$\frac{1}{A} [u] = \frac{c_v}{AR} \left[\frac{p}{\rho}\right] = \frac{1}{k-1} \left[\frac{p}{\rho}\right].$$

We then obtain in lieu of (1.44)

$$(p_+ + p_-) \left[\frac{1}{\rho}\right] + \frac{2}{k-1} \left[\frac{p}{\rho}\right] = 0.$$

Writing out the discontinuities in full, we obtain

$$p_+ \left(\frac{1}{\rho_+} - \frac{1}{\rho_-}\right) + \frac{2}{k-1} \frac{p_+}{\rho_+} = \frac{2}{k-1} \frac{p_-}{\rho_-} - p_- \left(\frac{1}{\rho_+} - \frac{1}{\rho_-}\right),$$

$$p_+ \{\rho_-(k-1) - p_+(k-1) + 2p_+\} =$$
$$= p_- \{2p_+ - (k-1)\rho_- + (k-1)\rho_+\},$$

or, ultimately,

$$\frac{p_-}{p_+} = \frac{(k+1)\rho_- - (k-1)p_+}{(k+1)p_+ - (k-1)\rho_-}. \qquad (1.45)$$

This is the equation of the dynamic adiabatic curve of an ideal gas, frequently also called the Hugoniot adiabatic curve.

Let us compare the dynamic adiabatic curve with the Poisson adiabatic curve:

$$\frac{p_-}{p_+} = \left(\frac{\rho_-}{\rho_+}\right)^k. \qquad (1.46)$$

The curves intersect at a point with coordinates (1; 1) (Fig. 6).
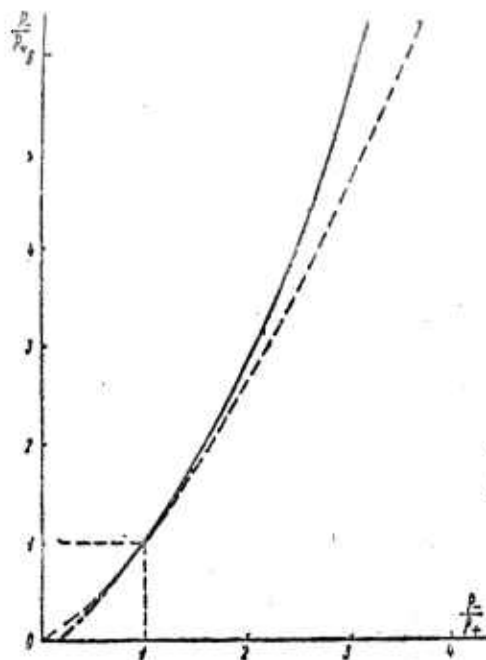
Fig. 6. Comparison of the dynamic adiabatic curve (solid) and Poisson adiabatic curve (dashed).

In the vicinity of this point we can put $\rho_- = \rho_+ + \Delta\rho$.

We then obtain in accordance with (1.46)

$$\frac{p_-}{p_+} = \left(1 + \frac{\Delta\rho}{\rho_+}\right)^k = 1 + k\,\frac{\Delta\rho}{\rho_+} + \frac{k(k-1)}{1\cdot2}\left(\frac{\Delta\rho}{\rho_+}\right)^2 +$$
$$+ \frac{k(k-1)(k-2)}{1\cdot2\cdot3}\left(\frac{\Delta\rho}{\rho_+}\right)^3 + \ldots \qquad (1.47)$$

For the shock adiabatic curve we have at this point

$$\frac{p_-}{p_+} = \frac{(k+1)(\rho_+ + \Delta\rho) - (k-1)\rho_+}{(k+1)\rho_+ - (k-1)(\rho_+ + \Delta\rho)} = \frac{2\rho_+ + (k+1)\Delta\rho}{2\rho_+ - (k-1)\Delta\rho} =$$
$$= 1 + k\,\frac{\Delta\rho}{\rho_+} + \frac{k(k-1)}{1\cdot2}\left(\frac{\Delta\rho}{\rho_+}\right)^2 + \frac{k(k-1)^2}{4}\left(\frac{\Delta\rho}{\rho_+}\right)^3 + \ldots \qquad (1.48)$$

It is easy to note that the foregoing expansions differ from each other only starting with the terms that contain $(\Delta\rho/\rho_+)^3$.

It follows therefore that the dynamic adiabatic curve and the Poisson adiabatic curve are not only tangent at the point $p_-/p_+ = 1$, $\rho_-/\rho_+ = 1$, but have also the same curvature at this point.

Unlike the Poisson adiabatic curve, the ratio p_-/p_+ in the shock adiabatic curve (1.45) vanishes not for $\rho_-/\rho_+ = 0$, but for $\rho_-/\rho_+ = 1/6$.

When $k = 1.4$ and $\rho_-/\rho_+ = 6$, we get $p_-/p_+ = \infty$.

Thus, on a strong-discontinuity surface at large compression pressures, the density increases relatively slowly, corresponding to a rapid growth of the ratio p/ρ, which determines the gas temperature.

Of great importance is also the fact that the equation of dynamic ·adiabatic curve (1.45) cannot be written in the form* $f(p_+, \rho_+) = f(p_-, \rho_-)$.

Yet for the Poisson adiabatic curve, as follows from (1.46), we have

$$F(p_+, \rho_+) = F(p_-, \rho_-).$$

In order to cover all the Poisson-adiabatic curves, it is sufficient to run through a one-dimensional series of values of the entropy S, and in order to cover all the dynamic-adiabatic curves, it is necessary to construct an "infinity squared" set of curves corresponding to all possible values of p_+ and ρ_+. In other words, unlike the Poisson adiabatic curve, along which the entropy remains constant, each point of the dynamic adiabatic curve corresponds to one definite value of the entropy, pertinent to this point only.

Consequently, if a region of variable pressure occurs behind the surface of a nonstationary strong discontinuity, then a region of variable entropy is simultaneously produced there.

As is well known, for an ideal gas the change in entropy is determined by the equation

$$\Delta S = \frac{c_p}{A} \ln \left\{ \frac{p_-}{p_+} \left(\frac{\rho_+}{\rho_-} \right)^k \right\}. \tag{1.49}$$

Let $\rho_- = \rho_+ + \Delta\rho$. Then after substituting the expansions (1.47) and (1.48) into (1.49) we obtain

$$\Delta S = -\frac{c_p}{A} \cdot \frac{k^2 - k}{12} \left(\frac{\Delta\rho}{\rho_+} \right)^3 \left[1 - \frac{3}{2} \frac{\Delta\rho}{\rho_+} \right]. \tag{1.50}$$

From this expression it follows that for small values of $\Delta\rho/\rho_+$

the increment of entropy in the shock wave is proportional to the cube
of the relative increase in density.
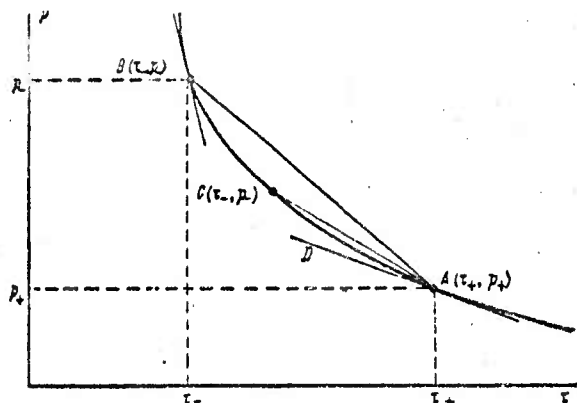
Fig. 7. Graphic comparison of
the propagation velocities of
shock waves and the local veloc-
ity of sound.

In weak shock waves the increment of entropy is so small that
without loss in general accuracy we can consider the propagation of
such waves as an isentropic process.

On the basis of a few other considerations we have arrived at a
similar conclusion previously, considering the adiabatic conditions
for water.

With the aid of the dynamic adiabatic curve we can readily estab-
lish the relation between the propagation velocities of shock waves
and the local velocity of sound. To this end we plot the dynamic adia-
batic curve in coordinates pressure (p) versus specific volume (τ)
(Fig. 7).

We reduce the propagation velocity θ to the same parameters.

We previously had

$$\theta_-^2 = \frac{p_+}{p_-} \frac{[p]}{[\rho]};$$ (1.42)

$$\theta_+^2 = \frac{p_-}{p_+} \frac{[p]}{[\rho]},$$ (1.43)

but

$$\frac{1}{\rho} = \tau.$$

Consequently, after elementary transformations we obtain

$$\theta_-^2 = \tau_-^2 \frac{p_- - p_+}{\tau_+ - \tau_-};$$

(1.51)

$$\theta_+^2 = \tau_+^2 \frac{p_- - p_+}{\tau_+ - \tau_-}.$$

(1.52)

Let us consider first Eq. (1.52).

For a given initial state of matter τ_+ and p_+, the factor τ_+^2 is a constant quantity and the propagation velocities of the shock waves θ_+, corresponding to different degrees of compression, depend on the ratio $(p_- - p_+)/(\tau_+ - \tau_-)$, i.e., on the slopes of the corresponding lines joining the initial point p_+, τ_+ with the points p_-, τ_-.

The rate of propagation of the perturbation at the point A will be determined by the slope of the tangent AD:

$$\theta_{+/A}^2 = -\tau^2 \frac{dp}{d\tau} = \frac{dp}{d\rho} = a_+^2$$

and obviously is equal to the velocity of sound a_+.

Thus, $\theta_+ > a_+$.

From perfectly analogous considerations (see Fig. 7) it also follows that $\theta_- < a_-$.

If we turn to stationary motions, for which $N = 0$ and $\theta = -v_n$, we obtain from the foregoing relations an important consequence: a strong discontinuity surface can occur only when the particle velocity v_n on one side of this surface exceeds the local velocity of sound. To distinguish it from shock waves, a strong-discontinuity surface that is stationary in space is called a compression shock.

In most cases of practical importance, the shock wave propagates in an unperturbed medium. Then $\theta_+ = N$ and $v_+ = 0$.

Thus, the displacement rate N of a shock wave always exceeds the velocity of sound ahead of the front, but the difference between the displacement rate of the shock wave and the particle velocity is always smaller than the velocity of sound behind the front, $N < a_- + v_{n-}$.

It follows therefore that weak perturbations which propagate in a liquid at the local velocity of sound, can catch up with the front of the shock wave but cannot overtake it.

We shall henceforth be interested principally in one-dimensional unsteady motion of liquids.

On the basis of the conditions for dynamic compatibility (1.39) and (1.40), we have in this case

$$\rho_0 (N - v_0) = \rho_\phi (N - v_\phi);$$

$$N = \frac{\rho_\phi v_\phi - \rho_0 v_0}{\rho_\phi - \rho_0};$$

$$v_\phi = N \frac{\rho_\phi - \rho_0}{\rho_\phi} + v_0 \frac{\rho_0}{\rho_\phi}; \qquad (1.53)$$

$$p_\phi - p_0 = \rho_0 (N - v_0)(v_\phi - v_0); \qquad (1.54)$$

$$(v_\phi - v_0)^2 = (p_\phi - p_0) \left(\frac{1}{\rho_0} - \frac{1}{\rho_\phi} \right). \qquad (1.55)$$

If $v_0 \equiv 0$, then

$$v_\phi = N \frac{\rho_\phi - \rho_0}{\rho_\phi}; \qquad (1.56)$$

$$p_\phi - p_0 = \rho_0 N v_\phi; \qquad (1.57)$$

$$v_\phi = \sqrt{(p_\phi - p_0) \left(\frac{1}{\rho_0} - \frac{1}{\rho_\phi} \right)}. \qquad (1.58)$$

The subscript "f" denotes here hydrodynamic parameters on the front of the wave, while the subscript "0" refers to the unperturbed liquid.

In this form, the relations between the hydrodynamic elements on the front of the shock wave are frequently used for practical calculations.

From the analysis of the dynamic adiabatic curve and from the second law of thermodynamics there follows an important physical consequence, known as the Zemplen theorem.

Zemplen's theorem states that the only strong discontinuities that are possible are those for which the pressure increases. No discontinuous rarefaction waves (rarefaction shocks) can exist.

Example 1. Establish from the conditions of dynamic compatibility

the connection between the value of the local velocity of sound $\underline{a}$ and the particle velocity $\underline{v}$ on the surface of a nonstationary strong discontinuity in an ideal gas. Assume that in the unperturbed medium $v_+ = = 0$, $p_+ = p_0$, $p_+ = \rho_0$ (N.Ye. Kochin, I.A. Kibel', N.V. Roze "Theoretical Hydrodynamics").

Solution. We choose for the initial equations

$$[\rho^0] = 0; \tag{1.39}$$

$$\rho^0[v_n] = [p]; \tag{1.40}$$

$$\frac{p_+}{p_-} = \frac{(k+1)\,p_+ - (k-1)\,p_-}{(k+1)\,p_- - (k-1)\,p_+}. \tag{1.45}$$

Leaving out the minus subscripts for the hydrodynamic elements on the discontinuity surface and taking into account the fact that $v_+ = 0$, we rewrite the system (1.39), (1.40), and (1.45) in the form

$$\rho_0 N v = p - p_0;$$

$$\rho_0 N = \rho\,(N - v);$$

$$\frac{p_0}{p} = \frac{(k+1)\,p_0 - (k-1)\,\rho}{(k+1)\,\rho - (k-1)\,p_0}.$$

Let us subtract unity from each half of the last equation. We then obtain

$$\frac{p_0 - p}{p} = \frac{[p]}{p} = \frac{(k+1)\,[p] + (k-1)\,[\rho]}{(k+1)\,\rho - (k-1)\,p_0} = \frac{2k\,[p]}{(k+1)\,\rho - (k-1)\,p_0}.$$

Consequently,

$$\frac{[p]}{[\rho]} = \frac{2kp}{(k+1)\,\rho - (k-1)\,p_0}.$$

But in accordance with (1.42), we have

$$c_-^2 = (N - v)^2 = \frac{p}{\rho}\cdot\frac{[p]}{[\rho]} = k\,\frac{p}{\rho}\,\frac{2p_0}{(k+1)\,\rho - (k-1)\,p_0} =$$

$$= a^2 \frac{2}{(k+1)\dfrac{\rho}{\rho_0} - (k-1)}.$$

Let us then write the equation of the dynamic adiabatic curve in the form

$$\frac{p}{p_0} = \frac{(k+1)\dfrac{\rho}{\rho_0} - (k-1)}{(k+1) - (k-1)\dfrac{\rho}{\rho_0}};$$

$$\frac{p - p_0}{p_0} = \frac{2k\left(\frac{\rho}{\rho_0} - 1\right)}{(k+1) - (k-1)\frac{\rho}{\rho_0}}.$$

Substituting in the result obtained the ratio ρ/ρ_0 from (1.39) and the difference $p - p_0$ from the equation given above, we obtain

$$\frac{\rho_0 N v}{p_0} = \frac{2k\left(\frac{N}{N-v} - 1\right)}{(k+1) - (k-1)\frac{N}{N-v}};$$

$$\frac{\rho_0 N}{k p_0} = \frac{2\frac{1}{N-v}}{(k+1) - (k-1)\frac{N}{N-v}};$$

$$\frac{N}{a_0^2} = \frac{2}{(N-v)(k+1) - (k-1)N}$$

or

$$N^2 - \frac{k+1}{2} N v - a_0^2 = 0. \tag{1.59}$$

Carrying out analogous substitutions and transformations with respect to Eq. (1.43), we get

$$N^2 + \frac{k-3}{2} N v - \frac{k-1}{2} v^2 - a^2 = 0. \tag{1.60}$$

Subtracting (1.60) from (1.59) we obtain

$$v N \left(\frac{k-3+k+1}{2}\right) + a_0^2 - a^2 - \frac{k-1}{2} v^2 = 0,$$

hence

$$N = \frac{v}{2} + \frac{a^2 - a_0^2}{k-1}\frac{1}{v}. \tag{1.61}$$

Substituting the results obtained in (1.59), we get

$$\frac{v^2}{4} + \frac{a^2 - a_0^2}{k-1} + \frac{(a^2 - a_0^2)^2}{(k-1)^2}\frac{1}{v^2} - \frac{k+1}{4} v^2 - \frac{k+1}{2(k-1)}(a^2 - a_0^2) - a_0^2 = 0,$$

or, ultimately,

$$v^4 + 2\frac{a^2 + a_0^2}{k} v^2 - \frac{4}{k}\frac{(a^2 - a_0^2)^2}{(k-1)^2} = 0. \tag{1.62}$$

The sought equation represents in the (a, v) plane a fourth-order curve (called hyposcissoid), which is symmetrical both about the $\underline{v}$ axis and about the $\underline{a}$ axis, with asymptotes $a = \pm\sqrt{\frac{k(k-1)}{2}}\, v$ (Fig. 8).

By virtue of Zemplen's theorem, we use in practice only that part

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of the curve for which a > a_0.
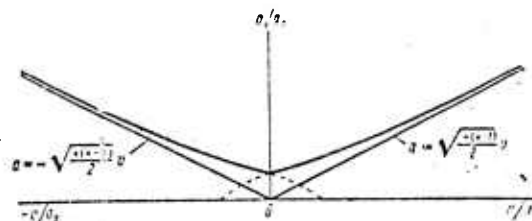
Fig. 8. Dependence of the local velocity of sound on the particle velocity.

Example 2. Express the hydrodynamic elements p, ρ, and v in an ideal gas on the surface of a nonstationary strong discontinuity in terms of the displacement velocity N.

Assume that the discontinuity surface propagates in an unperturbed medium $v_+ = 0$, $\rho = \rho_0$, $p = p_0$; $a = a_0$.

Solution. Let us use the equations established in the preceding problem:

$$N^2 - \frac{k+1}{2} Nv - a_0^2 = 0;$$ (1.59)

$$\frac{\rho}{\rho_0} = \frac{N}{N-v};$$ (1.40a)

$$p - p_0 = \rho_0 Nv,$$ (1.57)

On the basis of the first of the relations written out we have

$$v = \frac{2}{k+1} \frac{N^2 - a_0^2}{N}.$$

As a result

$$\frac{\rho}{\rho_0} = \frac{N}{N-v} = \frac{N}{N - \frac{2}{k+1} \frac{N^2 - a_0^2}{N}} = \frac{N^2}{\frac{k-1}{k+1} N^2 + \frac{2}{k+1} a_0^2};$$ (1.63)

$$p - p_0 = \rho_0 Nv = \frac{2}{k+1} \rho_0 \left(N^2 - a_0^2\right).$$ (1.64)

§5. DIFFERENTIAL EQUATIONS OF GASDYNAMICS. THE BERNOULLI EQUATION

So far principal attention was paid to a clarification of the general laws governing the propagation of discontinuity surfaces. This, however, does not exhaust the description of the complicated character

of the unsteady motion of a liquid during an explosion.

The discontinuity of the hydrodynamic elements on the front of the shock wave is followed already by a continuous variation of these quantities, which is usually described by the system of differential equations of gasdynamics.

As is well known, these equations are: the Euler equation of motion; the continuity equation; the energy equation.

The Euler equation is a mathematical form of writing down the Do'alembert principle, according to which there exists at each instant an equilibrium between the forces acting on a liquid particle, including the inertia forces.

It has the form

$$\rho \frac{d\bar{v}}{dt} + \mathrm{grad}\, p = 0. \tag{1.65}$$

The value of the acceleration $d\bar{v}/dt$ can be represented as a sum of the local derivative of the velocity with respect to time $\partial\bar{v}/\partial t$ and the so-called convective derivative, characterizing the variation of the velocity in connection with the transition of the liquid particle from one point of space to another:

$$\frac{dv}{dt} = \frac{\partial\bar{v}}{\partial t} + (\bar{v}\nabla)\,\bar{v}, \tag{1.66}$$

where ∇ is the del operator, also called the Hamiltonian operator,

$$\nabla = \bar{i}\,\frac{\partial}{\partial x} + \bar{j}\,\frac{\partial}{\partial y} + \bar{k}\,\frac{\partial}{\partial z}.$$

We thus have

$$\frac{\partial\bar{v}}{\partial t} + (\bar{v}\nabla)\,\bar{v} + '\frac{1}{\rho}\,\mathrm{grad}\, p = 0. \tag{1.67}$$

In a rectangular coordinate system, Eq. (1.67) is equivalent to the system

$$\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial x},$$
$$\frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial y}, \qquad (1.68)$$
$$\frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial z}.$$

Let us use the transformation known from vector analysis:*

$$(\bar{v}\nabla)\bar{v} = \nabla \frac{v^2}{2} - \bar{v} \times \operatorname{rot} \bar{v}. \qquad (1.69)$$

We then obtain the equation of motion in the Gromeko form

$$\frac{\partial \bar{v}}{\partial t} + \nabla \frac{v^2}{2} - \bar{v} \times \operatorname{rot} \bar{v} + \frac{1}{\rho} \nabla p = 0. \qquad (1.70)$$

We introduce the function

$$\nabla P = \frac{1}{\rho} \nabla p. \qquad (1.71)$$

In the case of unsteady potential motion, in view of the vanish-
ing of the velocity-curl vector, curl $v = 0$, we have $v = \operatorname{grad} \varphi$. Conse-
quently,

$$\nabla \left(\frac{\partial \varphi}{\partial t} + \frac{v^2}{2} + P \right) = 0$$

or

$$\frac{\partial \varphi}{\partial t} + \frac{v^2}{2} + P = F(t). \qquad (1.72)$$

Expression (1.72) is known as the Lagrange-Cauchy integral.

In the case of stationary flow we have from the Gromeko equation

$$\nabla \left(\frac{v^2}{2} + P \right) = \bar{v} \times \operatorname{rot} \bar{v}. \qquad (1.73)$$

Multiplying both halves of (1.47) by $\bar{d}r = \bar{i}dx + \bar{j}dy + \bar{k}dz$, we ob-
tain

$$d \left(\frac{v^2}{2} + P \right) = \begin{vmatrix} dx & dy & dz, \\ v_x & v_y & v_z, \\ \Omega_x & \Omega_y & \Omega_z, \end{vmatrix} \qquad (1.74)$$

where Ω_x, Ω_y, and Ω_z are the components of the velocity-curl vector
rot $\bar{v}$.

The right half of (1.74) vanishes in one of the following cases:

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a) if we consider the motion along a stream line

$$\frac{dx}{v_x} = \frac{dy}{v_y} = \frac{dz}{v_z};$$ (1.75)

b) if we consider the motion along a vortex line

$$\frac{dx}{\Omega_x} = \frac{dy}{\Omega_y} = \frac{dz}{\Omega_z};$$ (1.76)

c) if the stream lines and vortex lines coincide

$$\frac{v_x}{\Omega_x} = \frac{v_y}{\Omega_y} = \frac{v_z}{\Omega_z};$$ (1.77)

d) if the motion is potential

$$\Omega_x = \Omega_y = \Omega_z = 0;$$

e) if the liquid is at rest

$$v_x = v_y = v_z = 0.$$

Then

$$P + \frac{v^2}{2} = \text{const.}$$ (1.78)

Equation (1.78) is known as the Bernoulli integral.

In view of the great variety of practically important applications of the Bernoulli integral, we shall recast it in the most frequently encountered form.

Recognizing that the isentropic motion of an ideal gas corresponds to the condition $p/\rho^k = \text{const}$, we obtain the function P in the form

$$P = \frac{k}{k-1} \frac{p}{\rho}.$$ (1.79)

For water, in accordance with Tait's equation, we have $(p + B)/\rho^n = \text{const}$, and consequently,

$$P = \frac{n}{n-1} \frac{p+B}{\rho}.$$ (1.80)

Thus, the Bernoulli integral can be represented as follows:
for an ideal gas

$$\frac{k}{k-1} \frac{p}{\rho} + \frac{v^2}{2} = \text{const},$$ (1.81)

for water

$$\frac{n}{n-1} \frac{p+B}{\rho} + \frac{v^2}{2} = \text{const.}$$ (1.82)

Alternately, recognizing that the velocity of sound in an ideal gas is $a^2 = kp/\rho$, we have

$$\frac{a^2}{k-1} + \frac{v^2}{2} = \text{const.}$$ (1.83)

The value of the velocity of sound at a point where the particle velocity is equal to zero is usually designated by a*. Frequently, one uses also the concept of the critical velocity, which is the gas velocity equal to the local velocity of sound ($v_{kr} = a_{kr}$).

With the aid of the notation employed, we can write the Bernoulli integral in the form

$$\frac{a^2}{k-1} + \frac{v^2}{2} = \frac{a^{*2}}{k-1},$$ (1.84)

$$\frac{a^2}{k-1} + \frac{v^2}{2} = \frac{k+1}{k-1} \frac{a_{kp}^2}{2},$$ (1.85)

with

$$a_{kp} = \sqrt{\frac{2}{k+1}} \, a^*.$$ (1.86)

Relations of similar character hold true also for water, except that the isentropic exponent k must be replaced by the coefficient n:

$$\frac{a^2}{n-1} + \frac{v^2}{2} = \frac{a^{*2}}{n-1} = \frac{n+1}{n-1} \frac{a_{kp}^2}{2}.$$ (1.87)

Let us return to the fundamental differential equations of gasdynamics.

The mathematical form for writing down the law of mass conservation is the continuity equation

$$\frac{\partial \rho}{\partial t} + \text{div}\,(\rho \vec{v}) = 0,$$ (1.88)

or, if we write out the divergence in full,

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho v_x)}{\partial x} + \frac{\partial (\rho v_y)}{\partial y} + \frac{\partial (\rho v_z)}{\partial z} = 0.$$ (1.89)

We shall henceforth have to consider most frequently one-dimensional motion of liquids.

This is defined as liquid motion that depends on two coordinates: the space coordinate r and the time t. Particular cases of one-dimensional motion will be motions with plane, cylindrical, and spherical symmetry.

Obviously, the Euler equation for one-dimensional motion can be written in the form

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial r} + \frac{1}{\rho} \frac{\partial p}{\partial r} = 0. \qquad (1.90)$$

The continuity equation for such a motion can be written in the form

$$\frac{\partial \rho}{\partial t} + v \frac{\partial \rho}{\partial r} + \rho \frac{\partial v}{\partial r} + \frac{(\nu - 1)\rho v}{r} = 0, \qquad (1.91)$$

where $\nu = 1$, 2, or 3 for motion with plane, cylindrical, and spherical symmetry, respectively.

Since we are considering the motion of a liquid that has no viscosity or heat conduction, the entropy of any particle of the liquid remains constant, i.e.,

$$dS/dt = 0$$

or

$$\frac{\partial S}{\partial t} + v \frac{dS}{dr} = 0. \qquad (1.92)$$

Bearing in mind that $p/\rho^k = \varphi(S)$ for an ideal gas and $(p + C)/p^* = (\rho/\rho^*)^{\kappa(S)}$ for water, the adiabatic condition (1.92) can accordingly be written in the form

$$\frac{\partial}{\partial t} \ln\left(\frac{p}{\rho^k}\right) + v \frac{\partial}{\partial r} \ln\left(\frac{p}{\rho^k}\right) = 0; \qquad (1.93)$$

$$\frac{\partial}{\partial t}\left(\frac{p}{\rho^k}\right) + v \frac{\partial}{\partial r}\left(\frac{p}{\rho^k}\right) = 0; \qquad (1.94)$$

$$\frac{\partial}{\partial t}\left\{\left(\frac{p+C}{p^*}\right)^{\frac{1}{\ln\left(\frac{\rho}{\rho^*}\right)}}\right\} + v \frac{\partial}{\partial r}\left\{\left(\frac{p+C}{p^*}\right)^{\frac{1}{\ln\left(\frac{\rho}{\rho^*}\right)}}\right\} = 0. \qquad (1.95)$$

Equations (1.93; 1.94) or else (1.95), together with the equation of motion (1.90) and the continuity equation (1.91) form a closed system.

There exist no general methods for an exact solution of such a system.

For an approximate analysis of such systems one usually employs the method of characteristics, the gist of which is explained in the next section.

§6. EQUATIONS OF THE CHARACTERISTICS OF ONE-DIMENSIONAL UNSTEADY MOTION OF A LIQUID

Let us consider the system of differential equations of one-dimensional motion of an ideal gas

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial r} + \frac{1}{\rho} \frac{\partial p}{\partial r} = 0; \qquad (1.90)$$

$$\frac{\partial \rho}{\partial t} + v \frac{\partial \rho}{\partial r} + \rho \frac{\partial v}{\partial r} + \frac{(\nu - 1)\rho v}{r} = 0; \qquad (1.91)$$

$$\frac{\partial}{\partial t} \ln \frac{p}{\rho^k} + v \frac{\partial}{\partial r} \ln \frac{p}{\rho^k} = 0. \qquad (1.93)$$

If we assume that

$$\frac{p}{\rho^k} = \theta^k. \qquad (1.96)$$

In Eqs. (1.90), (1.91), and (1.93) we change over to new variables $\underline{v}$, $\underline{a}$, and θ. We note that

$$a^2 = \frac{dp}{d\rho} = \frac{kp}{\rho} = k\theta^k \rho^{k-1} = k\theta p^{\frac{k-1}{k}},$$

hence

$$\ln \rho = \frac{2}{k-1} \ln a - \frac{1}{k-1} \ln \theta^k - \frac{1}{k-1} \ln k;$$

$$\ln p = \frac{2k}{k-1} \ln a - \frac{k}{k-1} \ln \theta - \frac{k}{k-1} \ln k.$$

Consequently,

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial r} = - \frac{kp}{\rho} \frac{1}{kp} \frac{\partial p}{\partial r} = - \frac{a^2}{k} \frac{\partial \ln p}{\partial r} =$$

$$= - \frac{a^2}{k} \frac{2k}{k-1} \frac{\partial \ln a}{\partial r} + \frac{a^2}{k} \frac{k}{k-1} \frac{\partial \ln \theta}{\partial r} =$$

$$= -\frac{2a}{k-1}\frac{\partial a}{\partial r} + \frac{a^2}{k-1}\frac{\partial \ln \theta}{\partial r}.$$

Thus, in the new variables the equation of motion assumes the form

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial r} = -\frac{2a}{k-1}\frac{\partial a}{\partial r} + \frac{a^2}{k-1}\frac{\partial \ln \theta}{\partial r}. \qquad (1.97)$$

Using the expression for $\ln \rho$, we rewrite the continuity equation in the form

$$\frac{\partial \ln \rho}{\partial t} + v\frac{\partial \ln \rho}{\partial r} + \frac{\partial v}{\partial r} + \frac{(\nu-1)v}{r} = 0;$$

$$\frac{2}{k-1}\frac{1}{a}\Big(\frac{\partial a}{\partial t} + v\frac{\partial a}{\partial r}\Big) - \frac{1}{k-1}\Big(\frac{\partial \ln \theta^k}{\partial t} + v\frac{\partial \ln \theta^k}{\partial r}\Big) +$$

$$+ \frac{\partial v}{\partial r} + \frac{(\nu-1)v}{r} = 0.$$

But on the basis of the adiabaticity condition (1.93), the second bracket of the last equation vanishes, and consequently

$$\frac{\partial a}{\partial t} + v\frac{\partial a}{\partial r} + \frac{k-1}{2}a\frac{\partial v}{\partial r} + \frac{k-1}{2}\frac{(\nu-1)av}{r} = 0. \qquad (1.98)$$

Finally, the adiabaticity condition can be written in the form

$$\frac{\partial}{\partial t}\ln \theta + v\frac{\partial}{\partial r}\ln \theta = 0. \qquad (1.99)$$

Let us assume now that a curve L is specified on the r, t plane, $r = r(t)$, and that we know the functions v_L, a_L, θ_L on this curve (Fig. 9).

The problem consists of determining the integrals of the system of equations (1.97)-(1.99), which would assume specified values on the curve L. In other words, it is necessary to determine an integral surface satisfying Eqs. (1.97)-(1.99) and passing through the curve L.

The problem of determining the solution under such conditions is usually called the Cauchy problem.

We assume the functions v, a, and θ to be analytic in r and t.

On the curve L, the following relations are satisfied:
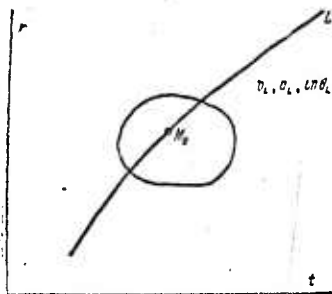
Fig. 9. Illustrating
the Cauchy problem.

$$\frac{\partial v_L}{\partial t} = \frac{\partial v_L}{\partial t} + r' \frac{\partial v_L}{\partial r},$$
$$\frac{\partial a_L}{\partial t} = \frac{\partial a_L}{\partial t} + r' \frac{\partial a_L}{\partial r}, \qquad (1.100)$$
$$\frac{\partial \ln \theta_L}{\partial t} = \frac{\partial \ln \theta_L}{\partial t} + r' \frac{\partial \ln \theta_L}{\partial r}.$$

At the same time, the hydrodynamic equations (1.97)-(1.99) must be satisfied on the same curve.

Let us ascertain what limitations are imposed on the fundamental system of equations by the fact that the solution of the system must correspond to specified values of the hydrodynamic elements on the curve L. To this end we obtain from (1.100) the values of the partial derivatives $\partial v_L/\partial t$, $\partial a_L/\partial t$, and $(\partial \ln \theta_L)/\partial t$, and substitute them in Eqs. (1.97)-(1.99).

We obtain

$$\frac{\partial v_L}{\partial t} = \frac{\partial v_L}{\partial t} - r' \frac{\partial v_L}{\partial r},$$
$$\frac{\partial a_L}{\partial t} = \frac{\partial a_L}{\partial t} - r' \frac{\partial a_L}{\partial r}, \qquad (1.101)$$
$$\frac{\partial \ln \theta_L}{\partial t} = \frac{\partial \ln \theta_L}{\partial t} - r' \frac{\partial \ln \theta_L}{\partial r};$$

$$(v - r') \frac{\partial v}{\partial r} + \frac{2a}{k-1} \frac{\partial a}{\partial r} - \frac{a^2}{k-1} \frac{\partial \ln \theta}{\partial t} = - \frac{\partial v_L}{\partial t}; \qquad (1.102)$$

$$\frac{k-1}{2} a \frac{\partial v}{\partial r} + (v - r') \frac{\partial a}{\partial r} = - \frac{\partial a_L}{\partial t} - \frac{k-1}{2} \frac{(\nu-1)}{r} av; \qquad (1.103)$$

$$(v - r') \frac{\partial \ln \theta}{\partial r} = - \frac{\partial \ln \theta_L}{\partial t}. \qquad (1.104)$$

The system of equations (1.102)-(1.104) enables us, generally speaking, to determine the partial derivatives $\partial v/\partial r$, $\partial a/\partial r$, and $(\partial \ln \theta)/\partial r$.

At the same time

$$\frac{\partial v}{\partial r} = \frac{\Delta_1}{\Delta}; \quad \frac{\partial a}{\partial r} = \frac{\Delta_2}{\Delta}; \quad \frac{\partial \ln \theta}{\partial r} = \frac{\Delta_3}{\Delta},$$

where Δ are the corresponding determinants of the system (1.102)-(1.104).

During the course of estimating the values of the partial derivatives $\partial v/\partial r$, $\partial a/\partial r$, and $(\partial \ln \theta)/\partial r$, the following principal cases are

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possible:

$$1. \quad \Delta \neq 0; \qquad \Delta_1 \neq 0; \qquad \Delta_2 \neq 0; \qquad \Delta_3 \neq 0$$
$$2. \quad \Delta = 0; \qquad \Delta_1 \neq 0; \qquad \Delta_2 \neq 0; \qquad \Delta_3 \neq 0$$
$$3. \quad \Delta = 0; \qquad \Delta_1 = 0; \qquad \Delta_2 = 0; \qquad \Delta_3 = 0$$

In the first case we obtain perfectly defined values of the derivatives at the point M_0 and its vicinity along the curve L. The Cauchy problem has a unique solution.

In the second case the values of the derivatives are infinite. The curve L in the vicinity of the point M_0 is called a "discontinuity line." In the third case there exists an infinite number of bodies of the derivatives $\partial v/\partial r$, $\partial a/\partial r$, and $(\partial \ln \theta)/\partial r$, corresponding to the given problem. In other words, it is possible to draw through the curve L in the vicinity of the point M_0 an infinite set of integral surfaces. Then the Cauchy problem has no unique solution. The direction of the tangent to the curve L at the point M_0 is called the characteristic direction.

The curve L, at each point of which the direction of the tangent corresponds to the characteristic equation, is called the characteristic. In the general theory of differential equations it is proved that:

1) the solution of the equations of the characteristics is equivalent to the solution of the corresponding initial system of equations;

2) for any change of variables, establishing a unique transition from the points of one space into the points of another space, the characteristics of the given equation go over into the characteristics of the transformed equations.

As follows from the definition, the equations of the characteristics can be readily written out by equating to zero the determinants of the initial system.

We then obtain ordinary differential equations, the analysis of which is much simpler than that of the initial system.

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Let us carry out such transformations for Eqs. (1.102)-(1.104).
As a result we obtain

$$\Delta = \begin{vmatrix} v-r' & \dfrac{2a}{k-1} & -\dfrac{a^2}{k-1} \\ \dfrac{k-1}{2}a & v-r' & 0 \\ 0 & 0 & v-r' \end{vmatrix} \qquad (1.105)$$

$$\Delta_1 = \begin{vmatrix} -\dfrac{dv_L}{dt} & \dfrac{2a}{k-1} & -\dfrac{a^2}{k-1} \\ -\dfrac{da_L}{dt} - \dfrac{(k-1)}{2}\dfrac{(v-1)av}{r} & v-r' & 0 \\ -\dfrac{d\ln\theta_L}{dt} & 0 & v-r' \end{vmatrix} \qquad (1.106)$$

$$\Delta_2 = \begin{vmatrix} v-r' & -\dfrac{dv_L}{dt} & -\dfrac{a^2}{k-1} \\ \dfrac{k-1}{2}a & -\dfrac{da_L}{dt} - \dfrac{k-1}{2}\dfrac{(v-1)av}{r} & 0 \\ 0 & -\dfrac{d\ln\theta_L}{dt} & v-r' \end{vmatrix} \qquad (1.107)$$

$$\Delta_3 = \begin{vmatrix} v-r' & \dfrac{2a}{k-1} & -\dfrac{dv_L}{dt} \\ \dfrac{k-1}{2}a & v-r' & -\dfrac{da_L}{dt} - \dfrac{k-1}{2}\dfrac{(v-1)av}{r} \\ 0 & 0 & -\dfrac{d\ln\theta_L}{dt} \end{vmatrix} \qquad (1.108)$$

In accordance with (1.105)

$$\Delta = (v-r')\,[(v-r')^2 - a^2] = 0. \qquad (1.109)$$

The determinant Δ vanishes in one of the following three cases:
a) $v - r' = 0$ or

$$\frac{dr}{dt} = v; \qquad (1.110)$$

in this case the equation of the characteristics coincides with
the differential equation of the trajectory of the liquid particles;
b) $v - r' - a = 0$;

$$\frac{dr}{dt} = v - a. \qquad (1.111)$$

c) $v - r' + a = 0$;

$$\frac{dr}{dt} = v + a. \qquad (1.112)$$

From Eqs. (1.111) and (1.112) it follows that the characteristics
propagate with the velocity of sound.

Let us consider the determinant Δ_1:

$$\Delta_1 = -\frac{d \ln \theta_L}{dt} \frac{a^2}{k-1}(v-r') +$$

$$+ (v-r')\left\{-\frac{dv_L}{dt}(v-r') + \frac{2a}{k-1}\frac{da_L}{dt} + \frac{(v-1)a^2v}{r}\right\} = 0. \qquad (1.113)$$

a) When $v - r' = 0$, we have $\Delta_1 = 0$.

When $v - r' = \pm a$

$$-\frac{d \ln \theta_L}{dt}\frac{a^2}{k-1} \pm \frac{dv_L}{dt}a + \frac{2a}{k-1}\frac{da_L}{dt} + \frac{(v-1)a^2v}{r} = 0,$$

or

$$b) \quad \frac{dv_L}{dt} + \frac{2}{k-1}\frac{da_L}{dt} = \frac{a}{k-1}\frac{d \ln \theta_L}{dt} - \frac{(v-1)av}{r}; \qquad (1.114)$$

$$c) \quad \frac{dv_L}{dt} - \frac{2}{k-1}\frac{da_L}{dt} = -\frac{a}{k-1}\frac{d \ln \theta_L}{dt} + \frac{(v-1)av}{r}. \qquad (1.115)$$

Going through analogous derivations, we can easily show that from the vanishing of Δ_2 it follows that $\theta = \text{const}$ when $v - r' = 0$.

An analysis of the determinant Δ_3 does not impose any new conditions on the hydrodynamic elements.

We shall call the characteristic of the first family*

$$\frac{dr}{dt} = v + a. \qquad (1.116)$$

On this characteristic we have the condition

$$\frac{dv}{dt} + \frac{2}{k-1}\frac{da}{dt} = \frac{a}{k-1}\frac{d \ln \theta}{dt} - \frac{(v-1)'av}{r}. \qquad (1.117)$$

We shall assume the characteristic of the second family to be

$$\frac{dr}{dt} = v - a. \qquad (1.118)$$

On this characteristic the relation to be satisfied is

$$\frac{dv}{dt} - \frac{2}{k-1}\frac{da}{dt} = -\frac{a}{k-1}\frac{d \ln \theta}{dt} + \frac{(v-1)'av}{r}. \qquad (1.119)$$

Finally, the characteristic of the third family will be called

$$dr/dt = v. \qquad (1.120)$$

On this characteristic

$$\theta^k = \frac{p}{\rho^k} = \text{const}. \qquad (1.121)$$

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In the case of isentropic motion of a gas it is obviously suffi-
cient to consider only the characteristics of the first and second
families, and the conditions on these characteristics also simplify.

Namely, for the first family we have

$$\frac{dr}{dt} = v + a,$$
$$\frac{dv}{dt} + \frac{2}{k-1}\frac{da}{dt} = -\frac{(\gamma-1)\,av}{r}. \qquad (1.122)$$

For the second

$$\frac{dr}{dt} = v - a,$$
$$\frac{dv}{dt} - \frac{2}{k-1}\frac{da}{dt} = \frac{(\gamma-1)av}{r}. \qquad (1.123)$$

The systems of equations (1.122) and (1.123) cannot be solved in-
dependently of each other, since each represents two equations with
four unknowns.

Nonetheless, the problem of integrating these equations is essen-
tially simpler than the problem of integrating the initial partial dif-
ferential equations.

In this lies the main advantage of the method of characteristics.

The system (1.122)-(1.123) does not in general have integrable
combinations. However, at the present time there have been developed
effective graphoanalytic methods for solving the equations of the char-
acteristics, with specified degree of accuracy. In the particular case
of isentropic motion with plane symmetry, the equations of the charac-
teristics have first integrals. In this case it becomes possible to ob-
tain exact solutions, a brief familiarization with which constitutes
the scope of the next section.

Example 1. In the plane r, t we know the values of the hydrody-
namic elements of the liquid on a segment of the curve r = f(t). De-
termine the nonsteady motion with variable entropy in the region
bounded by the specified segment AB and two segments of the character-
istics of different families, drawn from the points A and B to the

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point of their intersection (N.Ye. Kochin, I.A. Kibel', and N.V. Roze "Theoretical Hydromechanics").

Solution. We make the approximate substitution

$$\frac{dr}{dt} \simeq \frac{\Delta r}{\Delta t};$$

$$\frac{dv}{dt} \simeq \frac{\Delta v}{\Delta t}.$$
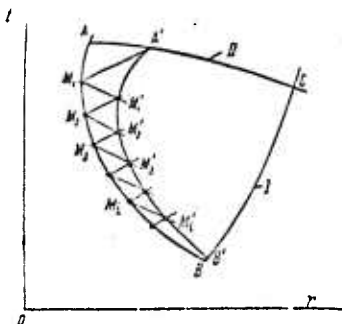
Fig. 10. Graphic illustration of the method of characteristics. I) First-family characteristics; II) second-family characteristics.

On the initial curve AB (Fig. 10) we mark a dense series of points M_1, M_2, ..., M_i.

Since we know the values of the hydrodynamic elements v and a at these points, the equations of the characteristics, written out in the form of finite-difference equations, constitute a system which when solved enables us to determine the elements of motion at the points of intersection of the characteristics of the different families M'_1, M'_2, M'_3, ..., M'_i.

Indeed, on the basis of (1.116)-(1.117) the equations of the characteristics of the first family will be

1) $r - r_i = (v + a)_i (t - t_i);$

2) $v - v_i + \frac{2}{k-1}(a - a_i) = \frac{a_i}{k-1}\ln\frac{\theta}{\theta_i} - \frac{(v-1)}{2}\left(\frac{av}{r} - \frac{a_i v_i}{r_i}\right).$

The equations of the characteristics of the second family will in accordance with (1.118) be

3) $r - r_{i+1} = (v - a)_{i+1} (t - t_1);$

4) $v - v_{i+1} - \frac{2}{k-1}(a - a_{i+1}) = -\frac{a_{i+1}}{k-1}\ln\frac{\theta}{\theta_{i+1}} + \frac{(v-1)}{2}\left(\frac{av}{2} - \frac{a_{i+1}v_{i+1}}{r_{i+1}}\right).$

If the entropy is constant $[\ln(\theta/\theta_{i+1}) = 0]$, then these four equations enable us to determine the points r, t, v, and a at the points M'_i.

In the case when we are studying motion with variable entropy, it is necessary to draw from the points M_i the third-family character-
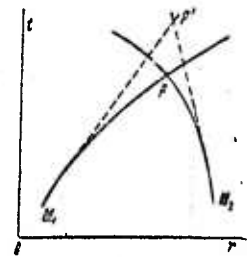
Fig. 11. Illustrating the method of characteristics (conditions of the problem).
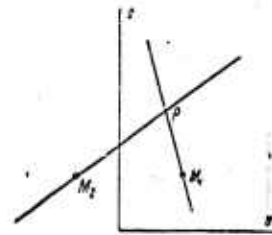
Fig. 12. Illustrating the method of characteristics (solution).

istics:*

    5) $r - r'_i = v(t - t'_i)$;

    6) $\theta = \theta'_i$;

    7) $r'_i = f(t'_i)$.

Again the number of equations is equal to the number of unknowns $(r, t, v, a, \theta, r'_i, t'_i)$.

Solving for each point M'_i an algebraic system of seven equations, we obtain the elements of motion at all points of the line A'B', which now can be regarded as the initial line for subsequent calculations.

The interval between the points M must be chosen from considerations of the permissible computation error.

Example 2. The values of v and a are given at two adjacent points M_1 and M_2 of the plane r, t. Find the values of v and a at the point P where the characteristics of different families, drawn from M_1 and M_2, intersect, for the case of isentropic motion with plane symmetry (Fig. 11).

Solution. Transform to the plane v, a. Knowing the values of v_{M_1}, a_{M_1} and v_{M_2}, a_{M_2}, we obtain the points M_1 and M_2 in this plane and draw the corresponding characteristics of different families. The coordinates of the point of intersection determine the sought-for values

of v and a at the point P (Fig. 12).

The position of the point P in the t, r plane is obtained approximately by replacing the characteristic lines by segments of tangents drawn through M_1 and M_2.

We have

$$r - r_1 = (v + a) M_1 (t - t_1); \qquad r - r_2 = (v - a) M_2 (t - t_2),$$

from which we obtain the values of r and t.

§7. SOME EXACT SOLUTIONS FOR A ONE-DIMENSIONAL UNSTEADY ISENTROPIC MOTION OF A LIQUID WITH PLANE SYMMETRY

In one-dimensional motion with plane symmetry ($v = 1$), the equations of the characteristics assume the following form in accordance with (1.122) and (1.123):

$$\left.\begin{aligned} \frac{dr}{dt} &= v + a, \\ v + \frac{2}{k-1} a &= \text{const} = 2\xi; \end{aligned}\right\} \tag{1.124}$$

$$\left.\begin{aligned} \frac{dr}{dt} &= v - a, \\ v - \frac{2}{k-1} a &= \text{const} = 2\eta. \end{aligned}\right\} \tag{1.125}$$

We shall regard r and t as functions of the characteristic numbers ξ and η

$$\left.\begin{aligned} r &= r(\xi, \eta), \\ t &= t(\xi, \eta). \end{aligned}\right\} \tag{1.126}$$

Since ξ is constant on the characteristic of the first family, the values of r and t on this characteristic will be functions of η only

$$dr = \frac{\partial r}{\partial \eta} d\eta; \quad dt = \frac{\partial t}{\partial \eta} d\eta.$$

It follows therefore that on the characteristic of the first family

$$\frac{\partial r}{\partial \eta} = (v + a) \frac{\partial t}{\partial \eta}. \tag{1.127}$$

We obtain quite analogously on the characteristic of the second family

$$\frac{\partial r}{\partial \xi} = (v - a) \frac{\partial t}{\partial \xi}, \tag{1.128}$$

but

$$v + \frac{2}{k-1} a = 2\xi; \quad v - \frac{2}{k-1} a = 2\eta.$$

It follows therefore that

$$\left. \begin{array}{l} v = \xi + \eta, \\ a = \frac{k-1}{2} (\xi - \eta). \end{array} \right\} \tag{1.129}$$

Substituting (1.129) in (1.127) and (1.128) we obtain ultimately

$$\left. \begin{array}{l} \frac{\partial r}{\partial \eta} = \left[\frac{k+1}{2} \xi + \frac{3-k}{2} \eta \right] \frac{\partial t}{\partial \eta}, \\ \frac{\partial r}{\partial \xi} = \left[\frac{3-k}{2} \xi + \frac{k+1}{2} \eta \right] \frac{\partial t}{\partial \xi}. \end{array} \right\} \tag{1.130}$$

The system (1.130) is called the canonical system of equations of gasdynamics for motion with constant entropy.

Unlike the quasilinear system (1.97) and (1.98), the system (1.130) is linear. It can be rewritten in an even simpler form. For this purpose we differentiate the first equation of the system (1.130) with respect to ξ and the second with respect to η.

As a result we obtain

$$\frac{\partial^2 r}{\partial \eta \partial \xi} = \frac{1}{2} [(k+1) \xi + (3-k) \eta] \frac{\partial^2 t}{\partial \xi \partial \eta} + \frac{\partial t}{\partial \eta} \frac{k+1}{2}; \tag{1.131}$$

$$\frac{\partial^2 r}{\partial \xi \partial \eta} = \frac{1}{2} [(3-k) \xi + (k+1) \eta] \frac{\partial^2 t}{\partial \xi \partial \eta} + \frac{\partial t}{\partial \xi} \frac{k+1}{2}. \tag{1.132}$$

Subtracting (1.132) from (1.131) we have

$$[(k-1) \xi - (k-1) \eta] \frac{\partial^2 t}{\partial \xi \partial \eta} + \frac{k+1}{2} \left(\frac{\partial t}{\partial \eta} - \frac{\partial t}{\partial \xi} \right) = 0,$$

or

$$\frac{\partial^2 t}{\partial \xi \partial \eta} = \frac{k+1}{2 (k-1) (\xi - \eta)} \left(\frac{\partial t}{\partial \xi} - \frac{\partial t}{\partial \eta} \right). \tag{1.133}$$

If $[(k + 1)/(k - 1)]/2 = m$, where m is an integer, Eq. (1.133) is called the Darboux equation and can be integrated in closed form.

In particular, for k = 1.4 we get

$$\frac{\partial^2 t}{\partial \xi \partial \eta} = \frac{3}{\xi - \eta} \left(\frac{\partial t}{\partial \xi} - \frac{\partial t}{\partial \eta} \right). \tag{1.134}$$

A general solution of (1.134) will be

$$t = \frac{\partial^4}{\partial \xi^2 \partial \eta^2} \frac{f_1(\xi) - f_2(\eta)}{\xi - \eta}.$$
(1.135)

The arbitrary functions $f_1(\xi)$ and $f_2(\eta)$ are obtained from the boundary conditions of the problem.

If the adiabatic exponent k exceeds three, the ratio $[(k + 1)/(k - 1)]/2$ can no longer be an integer. The question arises whether it is possible to construct exact solutions of a system of quasilinear equations of one-dimensional motion with plane symmetry, if one assumes a different form for the dependence of the density on the pressure, which has, however, a sufficient number of free parameters which can be managed at one's discretion. In other words, do there exist any forms of an approximate representation $p = f(\rho)$, permitting integration in closed form of the system:

$$\begin{cases} \frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} + \frac{1}{\rho} \frac{\partial p}{\partial x} = 0; \\ \frac{\partial \rho}{\partial t} + v \frac{\partial \rho}{\partial x} + \rho \frac{\partial v}{\partial x} = 0. \end{cases}$$
(1.136)

Let us show that such forms of approximate representation actually exist.

Let the isentropic condition be given by the relation

$$p = f(\rho),$$
(1.137)

and then the system (1.136) can be rewritten in the form

$$\begin{cases} \frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} + \frac{1}{\rho} f'(\rho) \frac{\partial \rho}{\partial x} = 0, \\ \frac{\partial \rho}{\partial t} + v \frac{\partial \rho}{\partial x} + \rho \frac{\partial v}{\partial x} = 0. \end{cases}$$
(1.138)

Assuming on the characteristics

$$\begin{aligned} \frac{\partial v}{\partial t} &= \frac{dv}{dt} - \frac{\partial v}{\partial x} \frac{dx}{dt}, \\ \frac{\partial \rho}{\partial t} &= \frac{d\rho}{dt} - \frac{\partial \rho}{\partial x} \frac{dx}{dt}, \end{aligned}$$
(1.139)

we obtain from (1.139) and (1.138) after substituting $\partial v/\partial t$ and $\partial \rho/\partial t$

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$$\frac{\partial v}{\partial x}\left(v - \frac{dx}{dt}\right) + \frac{1}{\rho}f'(\rho)\frac{\partial \rho}{\partial x} = -\frac{dv}{dt}, \\ \frac{\partial v}{\partial x}\rho + \left(v - \frac{dx}{dt}\right)\frac{\partial \rho}{\partial x} = -\frac{d\rho}{dt}. \quad (1.140)$$

From the condition that the main determinant of the system (1.140) vanish, we have

$$\frac{dx}{dt} = v \pm \sqrt{f'(\rho)}. \quad (1.141)$$

From the condition of the vanishing of the first determinant of the system (1.140) we get

$$-\frac{dv}{dt}\left(v - \frac{dx}{dt}\right) - \frac{d\rho}{dt}\frac{1}{\rho}f'(\rho) = 0$$

or, after substituting (1.141) and integrating,

$$v \pm \int_0^\rho \frac{1}{\rho}\sqrt{f'(\rho)}\, d\rho = \text{const.} \quad (1.142)$$

Thus, the equations of the characteristics of the first family will be

$$dx = \left(v + \sqrt{f'(\rho)}\right)dt; \\ v + \int_0^\rho \frac{1}{\rho}\sqrt{f'(\rho)}\, d\rho = 2\xi; \quad (1.143)$$

and those of the second family

$$dx = \left(v - \sqrt{f'(\rho)}\right)dt, \\ v - \int_0^\rho \frac{1}{\rho}\sqrt{f'(\rho)}\, d\rho = 2\eta. \quad (1.144)$$

Bearing in mind that ξ is constant on the characteristics of the first family and η is constant on the characteristics of the second family, we obtain by differentiation

$$\frac{\partial x}{\partial \eta} = \left(v + \sqrt{f'(\rho)}\right)\frac{\partial t}{\partial \eta}; \quad (1.145)$$

$$\frac{\partial x}{\partial \xi} = \left(v - \sqrt{f'(\rho)}\right)\frac{\partial t}{\partial \xi}, \quad (1.146)$$

with

$$v = \xi + \eta; \quad (1.147)$$

$$\int_0^\rho \frac{1}{\rho} V \overline{f'(\rho)}\, d\rho = \xi - \eta. \qquad (1.148)$$

We put

$$V \overline{f'(\rho)} = \varphi(\xi - \eta), \qquad (1.149)$$

and then

$$\frac{\partial x}{\partial \eta} = [(\xi + \eta) + \varphi(\xi - \eta)]\frac{\partial t}{\partial \eta}; \qquad \frac{\partial x}{\partial \xi} = [(\xi + \eta) - \varphi(\xi - \eta)]\frac{dt}{d\xi}.$$

Differentiating again, we have

$$\frac{\partial^2 x}{\partial \eta \partial \xi} = [1 + \varphi'(\xi - \eta)]\frac{\partial t}{\partial \eta} + [(\xi + \eta) + \varphi(\xi - \eta)]\frac{\partial^2 t}{\partial \eta \partial \xi}.$$

$$\frac{\partial^2 x}{\partial \xi \partial \eta} = [1 + \varphi'(\xi - \eta)]\frac{\partial t}{\partial \xi} + [(\xi + \eta) - \varphi(\xi - \eta)]\frac{\partial^2 t}{\partial \xi \partial \eta}.$$

Eliminating $\partial^2 x / \partial \xi \partial \eta$ from the last two equations, we obtain

$$\frac{\partial^2 t}{\partial \xi \partial \eta} + \frac{1}{2}\frac{1 + \varphi'(\xi - \eta)}{\varphi(\xi - \eta)}\left[\frac{\partial t}{\partial \eta} - \frac{\partial t}{\partial \xi}\right] = 0. \qquad (1.150)$$

Let the function $\nu(\xi - \eta)$ be such that the equation

$$\frac{\partial^2 t}{\partial \xi \partial \eta} + \nu(\xi - \eta)\left[\frac{\partial t}{\partial \eta} - \frac{\partial t}{\partial \xi}\right] = 0 \qquad (1.151)$$

can be integrated in closed form.

Then, putting for simplicity

$$\xi - \eta = \mu, \qquad (1.152)$$

we obtain $\varphi(\mu)$ without difficulty from the linear equation

$$\frac{1}{2}\frac{1 + \varphi'(\mu)}{\varphi(\mu)} = \nu(\mu), \qquad (1.153)$$

$$\varphi(\mu) = C_1 e^{\int 2\nu(\mu)\, d\mu} - e^{\int 2\nu(\mu)\, d\mu}\int e^{-\int 2\nu(\mu)\, d\mu}\, d\mu. \qquad (1.154)$$

Since, on the one hand, (1.148),

$$\int_0^\rho \frac{1}{\rho} V \overline{f'(\rho)}\, d\rho = \mu,$$

and on the other (1.149), $-\sqrt{f'(\rho)} = \varphi(\mu)$, the connection between μ and ρ is established with the aid of the equation

$$\frac{d\rho}{\rho} = \frac{d\mu}{\varphi(\mu)}. \qquad (1.155)$$

From the last relation we can determine $\mu(\rho)$ and consequently

also $\varphi[\mu(\rho)]$.

Recalling, finally, that $\sqrt{f'(\rho)} = \varphi(\mu) = \varphi[\mu(\rho)]$, we get

$$p = f(\rho) = \int \left\{ \varphi[\mu(\rho)] \right\}^2 d\rho + C_3. \qquad (1.156)$$

Thus, if we choose a certain function $\nu(\mu)$, then the connection between the pressure and the density is determined by a suitable functional relationship that includes three arbitrary constants, without counting those possible arbitrary constants which can enter into the function $\nu(\mu)$.

Thus, we have established a rather broad class of functions which permit integration of the main system of quasilinear equations (1.136) in closed form.

Let us consider some particular cases.

Let $\nu(\mu) = 0$, then

$$\frac{1}{2} \frac{1 + \varphi'(\mu)}{\varphi(\mu)} = 0,$$

hence

$$\varphi(\mu) = C_1 - \mu,$$
$$\int \frac{d\mu}{C_1 - \mu} = \ln C_2 \rho;$$
$$-\ln(C_1 - \mu) = \ln C_2 \rho.$$

Further

$$(C_1 - \mu) C_2 \rho = 1;$$
$$\mu = -\frac{1}{C_2 \rho} + C_1,$$

consequently,

$$\varphi(\mu) = \frac{1}{C_2 \rho}; \quad \sqrt{f'(\rho)} = \frac{1}{C_2 \rho}.$$

For the pressure we obtain

$$f'(\rho) = \frac{1}{C_2^2 \rho^2}; \quad p = f(\rho) = C_3 - \frac{1}{C_1^2} \cdot \frac{1}{\rho}. \qquad (1.157)$$

This result, which is extensively used in stationary problems of gasdynamics, was indicated in its time by S.A. Chaplygin.

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The main system of equations can be integrated in closed form by replacing the real isentrope $p = f(\rho)$ in the plane $(p, \frac{1}{\rho})$ by a tangent drawn to it through some point.

The general integral of (1.150) will in this case be

with
$$t = \psi(\xi) + \chi(\eta),$$
$$\xi + \eta = v;$$
$$\xi - \eta = \frac{1}{C_2 \rho} + C.$$

We now put $\nu(\mu) = n/\mu$.

Then, as indicated above, Eq. (1.156) assumes the form of the so-called Darboux equation and is integrated in closed form if $n = 0, 1, 2, 3, \ldots$

We have

$$\frac{1}{2} \frac{1 + \varphi'(\mu)}{\varphi(\mu)} = \frac{n}{\mu}. \tag{1.158}$$

Integrating, we get

$$\varphi(\mu) = C_1 \mu^{2n} + \frac{1}{2n - 1} \mu. \tag{1.159}$$

First let $C_1 = 0$, and then $\varphi(\mu) = \frac{1}{2n - 1} \mu$;

$$\int \frac{d\mu}{\varphi(\mu)} = \int \frac{d\mu}{\mu} (2n - 1) = \ln C_2 \rho,$$

$$(2n - 1) \ln \mu = \ln C_2 \rho; \quad \mu = (C_2 \rho)^{\frac{1}{2n-1}};$$

$$\varphi[\mu(\rho)] = \frac{1}{2n - 1} (C_2 \rho)^{\frac{1}{2n-1}};$$

$$f'(\rho) = \{\varphi[\mu(\rho)]\}^2 = \left(\frac{1}{2n-1}\right)^2 C_2^{\frac{2}{2n-1}} \rho^{\frac{2}{2n-1}}.$$

Integrating the last equation, we obtain

$$f(\rho) = C_2 \rho^{\frac{2n+1}{2n-1}} + C_3.$$

Putting $(2n + 1)/(2n - 1) = k$, we arrive for this particular case likewise at a known result, which was indicated earlier: the system of

equations of a homogeneous unsteady motion of a liquid with plane symmetry can be integrated in closed form for an isentropic process if the dependence $p = f(\rho)$ is expressed by the polytrope $p = A\rho^k + B$, the exponent of which corresponds to the equation $[(k + 1)/(k - 1)]/2 = n$, where $\underline{n}$ is an integer.

Now let $C_1 \neq 0$.

We have

$$\varphi(\mu) = C_1\mu^{2n} + \frac{1}{2n-1}\mu; \qquad (1.159)$$

$$\int \frac{d\mu}{C_1\mu^{2n} - \frac{1}{2n-1}\mu} = \ln C_2\rho. \qquad (1.160)$$

The quadrature in the left half of (1.160) can be taken for the specified integer $\underline{n}$.

Since the solution of the Darboux equation has the simplest form when $n = 1$, we give the necessary derivations for this case:

$$\int \frac{d\mu}{C_1\mu^2 + \mu} = \ln C_2\rho;$$

$$-\ln \frac{C_1\mu + 1}{\mu} = \ln C_2\rho; \quad \frac{C_1\mu + 1}{\mu} = \frac{1}{C_2\rho};$$

$$\mu = \frac{C_2\rho}{1 - C_1C_2\rho}; \quad \varphi(\mu) = \frac{C_2\rho}{(1 - C_1C_2\rho)^2} = \sqrt{f'(\rho)};$$

$$f'(\rho) = \frac{C_2^2\rho^2}{(1 - C_1C_2\rho)^4};$$

$$p = f(\rho) = \left[\frac{\rho^2}{C_1C_2} - \frac{\rho}{(C_1C_2)^2} + \frac{1}{3(C_1C_2)^3}\right]\frac{C_2^2}{[1 - C_1C_2\rho]^3} + C_3. \qquad (1.161)$$

The presence of three parameters makes it possible to approximate the actual isentrope by means of Relation (1.161) with accuracy sufficient for practical purposes.

The general integral (1.161) for $v(\mu) = 1/\mu$ is determined by the relation

$$t = \frac{1}{\xi - \eta}[X(\xi) + \psi(\eta)]. \qquad (1.162)$$

Knowing $\underline{t}$ and using the general methods of analysis, we can readily obtain also an expression for $\underline{x}$.*

Let us stop to discuss one more particular case of isentropic motion of a liquid with plane symmetry, for which the equations of gasdynamics can be integrated in closed form.

Let the motion be such that the particle velocities v and the velocity of sound a remain constant along one of the characteristics of the first family in the r, t plane.

We previously obtained

$$\left.\begin{aligned} \frac{dr}{dt} &= v + a, \\ v + \frac{2}{k-1} a &= 2\xi; \end{aligned}\right\} \qquad (1.124)$$

$$\left.\begin{aligned} \frac{dr}{dt} &= v - a, \\ v - \frac{2}{k-1} a &= 2\eta. \end{aligned}\right\} \qquad \cdot \quad (1.125)$$

According to (1.124) such a characteristic will be a straight line, since $dr/dt = $ const on it.

Let us consider now the characteristics of the second family (Fig. 13), L_1, L_2, ..., L_i. Assume that they cross AB at the points M_1, M_2, ..., M_i.

The equations of these characteristics can be written in the form

$$v_{M_1} - \frac{2}{k-1} a_{M_1} = 2\eta_1;$$

$$v_{M_2} - \frac{2}{k-1} a_{M_2} = 2\eta_2;$$

$$\cdots \cdots \cdots \cdots$$
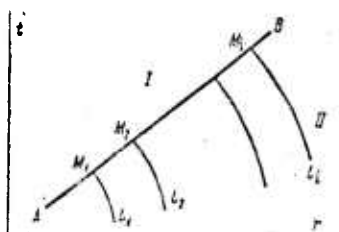
$$v_{M_n} - \frac{2}{k-1} a_{M_n} = 2\eta_n.$$

Fig. 13. Characteristics of second family in a straight wave in one direction.

But at the points M_1, M_2, ..., M_n, in accordance with the conditions of the problem, the values of v and a are constant. Consequently, the characteristic numbers η_i will also be constant.

Thus, the relation

$$v - \frac{2}{k-1} a = 2\eta_0 \qquad (1.163)*$$

will be satisfied not only along the specified characteristic, but

also in the entire plane.

It is appropriate to note that in the case under consideration it is not only one characteristic of the first family that is a straight line in the r, t plane, as was assumed previously, but in general all the first-family characteristics are straight lines.

Indeed, along each such characteristic we have in accord with (1.124) $v + \frac{2}{k-1} a = 2\xi$ (each characteristic having its own ξ), but, in addition, Eq. (1.163) is satisfied in the entire region.

Thus, the values of v and a on the first-family characteristics will be connected by two algebraic relations.

Thus the correctness of the statement formulated above follows from the first equation of (1.124).

We have already written down the equation of motion in the variables a and v:

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial r} + \frac{2}{k-1} a \frac{\partial a}{\partial r} = 0. \qquad (1.164)$$

Subtracting from (1.163) the quantity $a = \frac{k-1}{2} v - (k-1)\eta_0$ and substituting it in (1.164), we obtain

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial r} + \left[\frac{2}{k-1} \frac{k-1}{2} v - \frac{2(k-1)}{k-1} \eta_0 \right] \frac{k-1}{2} \frac{\partial v}{\partial r} =$$

$$= \frac{\partial v}{\partial t} + \left[\frac{k+1}{2} v - (k-1) \eta_0 \right] \frac{\partial v}{\partial r} = 0. \qquad (1.166)$$

This linear first-order differential equation is equivalent to the system of ordinary differential equations

$$\frac{dt}{1} = \frac{dr}{\frac{k+1}{2} v - (k-1)\eta_0} = \frac{dv}{0}, \qquad (1.167)$$

hence

$$v = \text{const} = C_1;$$

$$\left[\frac{k+1}{2} v - (k-1)\eta_0 \right] dt - dr = 0,$$

or

$$-\left[\frac{k+1}{2} v - (k-1)\eta_0 \right] t + r = C_2.$$

Finally, putting $C_2 = \varphi(C_1)$, we obtain ultimately

$$r = \left[\frac{k+1}{2} v - (k-1)\gamma_0\right] t + \varphi(v), \qquad (1.168)$$

and, in addition,

$$v - \frac{2}{k-1} a = 2\gamma_0. \qquad (1.163)$$

This result was first derived in 1860 by Riemann.

Solution (1.168)-(1.163) is called Riemann flow, or direct wave of one direction.

In perfectly analogous fashion, assuming that the particle velocities and the sound velocity remain constant on the characteristic of the second family, we will have

$$r = \left[\frac{k+1}{2} v - (k-1)\xi_0\right] t + \varphi_1(v); \qquad (1.169)$$

$$v + \frac{2}{k-1} a = 2\xi_0. \qquad (1.170)$$

Solution (1.169)-(1.170) is called the inverse wave of one direction.

As is well known, the general solutions of two first-order partial differential equations should depend on two arbitrary functions. It can be shown that the solutions obtained, which depend on only one arbitrary function, are not a particular case of a general solution but represent singular solutions, describing certain perfectly defined physical processes. The main property of these solutions is that the motions characterized by the direct and inverse waves of one direction can be contiguous with the region of rest, or, in the more general case, with the region of stationary motion of the medium.

We note that in the case when the function $\varphi(v)$ or $\varphi_1(v)$ vanishes identically, the motion of the liquid depends only on the ratio r/t. Then the distribution of the hydrodynamic elements at different instants of time will be similar to one another, differing only in the

scale along the r axis, which increases in proportion to the time. A natural generalization of such a motion is the case when the hydrodynamic elements are determined by the ratio r^α/t^β. Obviously, upon suitable change of the scales along the r and t axes (in particular, if logarithmic scales are used), it is possible to represent also in this case the pattern of the motion as if it remains similar to itself all the time. Motion of this type is called self-similar.

In self-similar motions it is always possible to reduce the problem of integrating the system of partial differential equations of gas-dynamics to the problem of integrating a system of ordinary differential equations.
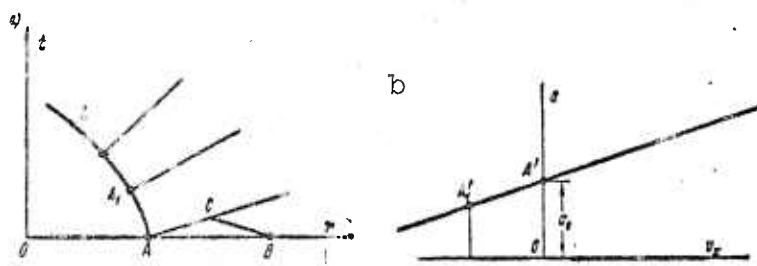
Fig. 14. Graphic illustration of the state of the gas in the case of a decelerated motion of a piston.

Example 1. A piston moves in an unbounded cylinder in accordance with a specified law r = f(t). At the initial instant of time the velocity of the piston is equal to zero and the gas is at rest.

Find the motion of the medium "to the right" of the piston, if

$$r = f(t) = -\frac{nt^2}{2}, \qquad (n > 0).$$

(Kochin, Kibel', Roze, Theoretical Hydromechanics, Vol. II).

Solution The fact that the medium is at rest at the initial instant of time denotes that v = 0 and the velocity of sound is a_0 = const.

Consequently, in the (r, t) plane the characteristics of the first family will be straight lines (Fig. 14a). The characteristic that sep-

arates the unperturbed medium from the region of motion is

$$\frac{dx}{dt} = a_0, \qquad x - x_0 = a_0 t.$$

In the (v, a) plane, on the basis of (1.125), the motion will be determined by the straight line (Fig. 14b)

$$v - \frac{2}{k-1} a = -\frac{2}{k-1} a_0.$$

The case under consideration corresponds to the singular solution

$$r = \left[\frac{k+1}{2} v - (k-1) \eta_0\right] t + \varphi(v), \qquad (1.168)$$

where

$$\eta_0 = -\frac{1}{k-1} a_0.$$

Consequently, Formula (1.169) can be written in the form

$$v = F\left[r - \left(\frac{k+1}{2} v + a_0\right) t\right].$$

The velocity of motion of the gas particles on the piston is obviously equal to the velocity of the piston:

$$f'(t) = F\left[r - \left(\frac{k+1}{2} f' + a_0\right) t\right].$$

But

$$f(t) = -\frac{nt^2}{2} \qquad f' = -nt,$$

therefore

$$-nt = F\left(-\frac{nt^2}{2} + \frac{k+1}{2} nt - a_0 t\right).$$

We put

$$-\frac{nt^2}{2} + \frac{k+1}{2} nt - a_0 t = \lambda.$$

The time t is determined from the quadratic equation

$$\frac{kn}{2} t^2 - a_0 t - \lambda = 0;$$

and consequently,

$$t = \frac{1}{kn}\left(a_0 + \sqrt{a_0^2 + 2kn\lambda}\right),$$

$$F(\lambda) = -\frac{1}{k}\left(a_0 + \sqrt{a_0^2 + 2kn\lambda}\right).$$

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Let us return to the previous variables, recognizing that on the piston we have $-nt^2/2 = r$, $-nt = v$. We obtain

$$\lambda = r - \left(\frac{k+1}{2} v + a_0\right) t,$$

$$F(\lambda) = -\frac{1}{k}\left\{a_0 + \sqrt{a_0^2 + 2kn\left[r - \left(\frac{k+1}{2} v + a_0\right) t\right]}\right\}.$$

Solving this equation with respect to v, we obtain ultimately

$$v = -\frac{1}{k}\left\{n\frac{k+1}{2}t + a_0 + \sqrt{\left[n\frac{(k+1)}{2}t + a_0^2\right] + 2nk(r - a_0 t)}\right\}.$$

The velocity of sound at the point with coordinates r, t is determined from the relation

$$v - \frac{2}{k-1} a = -\frac{2}{k-1} a_0.$$

If $v < 0$ and $|v| > 2a_0/(k-1)$, then the velocity of sound becomes negative and the solution loses its physical meaning. Vacuum is produced at the walls of the piston. If the law governing the motion of the piston is assumed to be $r = -nt^2/2$, this phenomenon sets in at the instant of time $t > 2a_0/(k-1)n$.

Example 2. Determine the instant of occurrence of a strong discontinuity surface in accelerated motion of a piston in an infinite cylinder.

The law of motion of the piston is $x = f(t)$. At the instant $t = 0$ the piston velocity is zero, and the medium is at rest (Kochin, Kibel', Roze, Theoretical Hydromechanics, Vol. II).

Solution. Since the motion of the piston occurs in an unperturbed medium, the characteristics of the first family will be straight lines (Fig. 15).

As in the first problem, the equation of the characteristic that separates the unperturbed medium from the region of motion will be $r - r_0 = a_0 t$.

In view of the fact that $f''(t)$ is positive (the motion of the pis-
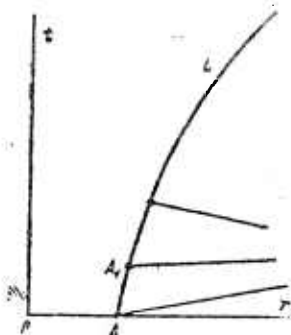
Fig. 15. Graphic
illustration of
the formation of
a strong-discon-
tinuity surface
in the accel-
erated motion of
a piston.

ton is accelerated), the velocities v of the particles on the piston increase with increasing time. The local velocity of sound a on the piston also increases:

$$\left(\frac{dx}{dt}\right)_{A_1} = (v)_{A_1} + a_{A_1} > 0 + a_0 = \left(\frac{dx}{dt}\right)_{A}.$$

The straight-line characteristics of the first family, emerging from various points L, gradually change their slope and begin to inter-sect sooner or later.

A strong discontinuity sets in (for more details see §7). Let us find the limit to which will tend the point of intersection of the characteristic drawn from the point A and the char-acteristic drawn from A_1 when A_1 tends to A.

For the first of these characteristics we have

$$x - x_0 = a_0 t;$$

for the second we have

$$x - x_0 - \Delta x = [a(x_0 + \Delta x_0,\ \Delta t) + v(x_0 + \Delta x;\ \Delta t)](t - \Delta t),$$

where Δx_0 is the difference between the abscissas of A_1 and A, and $\Delta t'$ is the difference between the ordinates of these points.

From this we obtain for the point of intersection of these lines

$$a_0 t - \Delta x_0 = [a + v](t - \Delta t),$$

or

$$t = \frac{-\Delta x + (a + v)\,\Delta t}{a + v - a_0},$$

but

$$\Delta x = f'(x_0)\,\Delta t,$$

$$v - \frac{2}{k - a} = -\frac{2}{k - 1} a_0,$$

hence

$$a = \frac{k - 1}{2}\, v + a_0.$$

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Consequently,

$$a + v - a_0 = \frac{k-1}{2} v + a_0 + v - a_0 = \frac{k+1}{2} v.$$

Let us expand the function v in a Taylor series in the vicinity of the point x_0:

$$v(x_0 + \Delta x, \Delta t) = v(x_0, 0) + \frac{dv(x_0, 0)}{dt} \Delta t + \ldots,$$

but

$$v(x_0, 0) = \frac{df}{dt}, \quad \left(\frac{\partial f}{\partial t}\right)_0 = 0,$$

$$a + v - a_0 = \frac{k+1}{2} v = \frac{k+1}{2} \frac{d^2 f}{dt^2} \Delta t.$$

The expression for t assumes the form

$$t = \frac{(a + v - f''(t)) \Delta t}{\frac{k+1}{2} \frac{d^2 f}{dt^2} \Delta t}.$$

Thus, going to the limit, we obtain the following expression for the instant of occurrence of the strong discontinuity:

$$\lim_{\Delta t \to 0} t = T = \frac{2a_0}{(k+1) f_0'}.$$

The position of the discontinuity surface relative to the origin will be

$$x = x_0 + a_0 T,$$

$$x = \frac{2}{k+1} \frac{a_0^2}{f_0'} + x_0.$$

Example 3. Approximate the isentropic equation of water in Tait's form by a relation of the type (1.161).

Solution. We previously had

$$p = f(\rho) = \left[\frac{\rho^2}{C_1 C_2} - \frac{\rho}{(C_1 C_2)^2} + \frac{1}{3(C_1 C_2)^3}\right] \frac{C_2^2}{[1 - C_1 C_2 \rho]^2} + C_3. \qquad (1.161)$$

According to Tait's equation

$$p + B = (p_0 + B)\left(\frac{\rho}{\rho_0}\right)^n. \qquad (1.22)$$

We choose for the approximation conditions the equality of the pressures and the sound velocities in the unperturbed medium:

$$\text{at } \rho = \rho_0 \begin{cases} p = p_0 \\ a = a_0 \end{cases}. \tag{1.171}$$

From a comparison of (1.161) and (1.22) with the condition (1.171) it follows immediately that the coefficient $C_3 = -B$.

Thus, the isentropic exponent $\underline{n}$ in Eq. (1.22) is replaced, as it were, by two coefficients C_1 and C_2, which are to be determined.

To evaluate these coefficients we have

$$p_0 + B = \left[\frac{\rho_0^2}{C_1 C_2} - \frac{\rho_0}{(C_1 C_2)^2} + \frac{1}{3(C_1 C_2)^3} \right] \frac{C_2^2}{[1 - C_1 C_2 \rho_0]^3} \tag{1.172}$$

and since

$$a = \sqrt{f'(\rho)} = \frac{C_2 \rho}{(1 - C_1 C_2 \rho)^2},$$

we have

$$a_0 = \frac{C_2 \rho_0}{(1 - C_1 C_2 \rho_0)^2}. \tag{1.173}$$

The coefficient C_2 can be readily eliminated from (1.161) with the aid of (1.173). We are then left with only the product $C_1 C_2 = x$ as the unknown quantity

$$p_0 + B = \left[\frac{\rho_0^2}{x} - \frac{\rho_0}{x^2} + \frac{1}{3x^3} \right] \frac{a_0^2 (1 - \rho_0 x)}{\rho_0^2},$$

or, what is the same,

$$\frac{p_0 + B}{\rho_0 a_0^2} = \left[\frac{1}{\rho_0 x} - \frac{1}{(\rho_0 x)^2} + \frac{1}{3(\rho_0 x)^3} \right] (1 - \rho_0 x),$$

But in accordance with (1.22) we have*

$$\rho_0 a_0^2 = (p_0 + B) n.$$

Thus, to determine the product $\rho_0 x = z$ we have the cubic equation

$$\frac{3z^2 - 3z + 1}{3z^3} (1 - z) = \frac{1}{n}, \tag{1.174}$$

the solution of which yields $z = 0.642$

The approximate relation is obtained in the form

$$\frac{p + B}{p_0 + B} = n \left[\left(\frac{\rho}{\rho_0} \right)^2 \frac{1}{z} - \left(\frac{\rho}{\rho_0} \right) \frac{1}{z^2} + \frac{1}{3} \frac{1}{z^3} \right] \frac{(1 - z)^4}{\left[1 - z \frac{\rho}{\rho_0} \right]^3}, \tag{1.175}$$

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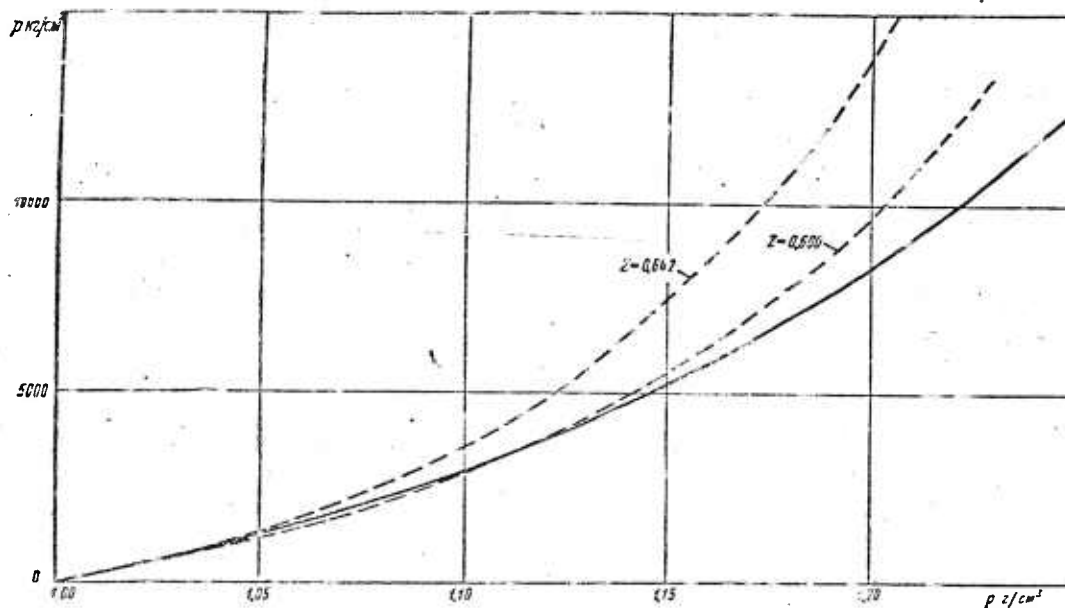
Fig. 16. Comparison of the initial (continuous line) and approximating (dash) isentropes in coordinates $\underline{p}$ and ρ.

or, after substituting the values of the coefficients

$$\frac{p+B}{p_0+B} = \left[1.557\left(\frac{\rho}{\rho_0}\right)^2 - 2{,}426\left(\frac{\rho}{\rho_0}\right) + 1{,}260\right]\frac{0{,}1171}{\left[1-0{,}642\left(\frac{\rho}{\rho_0}\right)\right]^3}. \qquad (1.176)$$

Figure 16 shows a comparison of the isentropes (1.176) and (1.22).
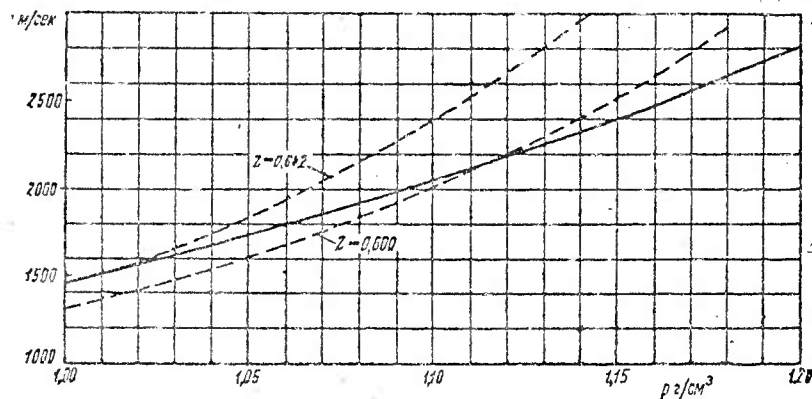
Fig. 17 Comparison of the initial (continuous line) and approximating (dash) isentropes in the coordinates $\underline{a}$ and ρ.

Satisfactory agreement is found in the region of pressures up to 2000 atm. Assuming the deviation of the velocity of sound in the unperturbed medium to be about 10% and putting z = 0.600, we can obtain a better approximation in the mean over a wider range of pressures.

The results of the corresponding calculations are shown in Fig. 17.
The approximating equation has the form

$$\frac{p+B}{p_0+B} = \left[1{,}667 \left(\frac{p}{p_0}\right)^2 - 2{,}777 \left(\frac{p}{p_0}\right) + 1{,}543 \right] \frac{0{,}1478}{\left[1 - 0{,}600 \frac{p}{p_0} \right]^3}. \qquad (1.177)$$

§8. PHYSICAL IDEAS CONCERNING SHOCK WAVES

So far we have examined the shock wave from the mathematical point of view, defining its front as a nonstationary strong-discontinuity surface.

Before we continue the exposition, it is important to make a few remarks concerning the physical aspect of the phenomenon.

We note that shock waves can occur not only as a result of explosions, but also in other physical processes.
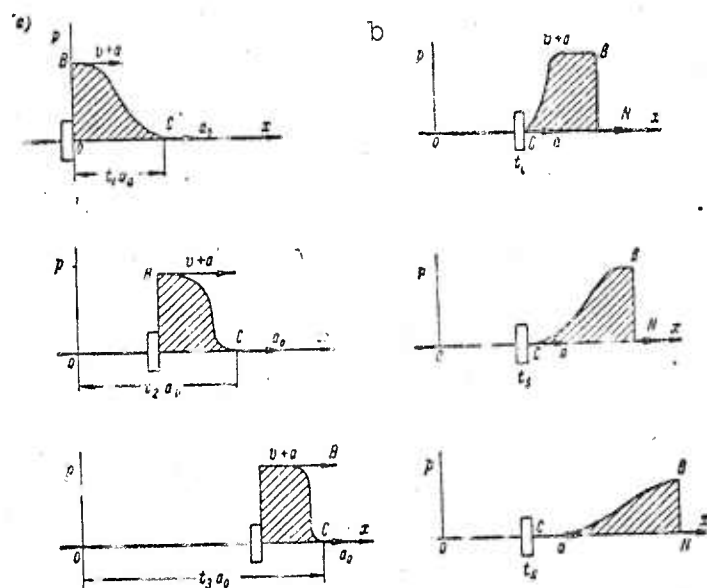
Fig. 18. Scheme of formation of the discontinuity surface and its collapse.

Indeed, if for example a piston moves with acceleration in an infinitely long tube, then the liquid directly in contact with its surface will all the time be compressed there to a greater degree than away from the piston, as a result of its inertial properties. The per-

turbations caused by the motion of the piston will thus propagate over a region that is characterized by a variable value of sound velocity. The local sound velocity will then monotonically decrease with increasing distance from the piston.

Consequently, the resultant perturbations will tend to catch up with the perturbations that are located ahead of them. This process will continue until a discontinuous front is produced, which we are accustomed to call the shock wave front (Fig. 18a). If the piston stops suddenly, then the region of compressed medium will continue to move to the right along the ox axis.

According to Zemplen's theorem, sudden stoppage of the piston cannot produce a negative pressure jump. Therefore a rarefaction wave is produced, which is an aggregate of elementary waves of pressure reduction, propagating with local velocity of sound. Some of these waves, which have a propagation velocity close to the local velocity of sound behind the front, will catch up at some definite instant of time with the front and reduce its amplitudes $(a + v > N)$. Another part, to the contrary, with a propagation velocity less than or equal to a_0, will lag the front. The pressure pattern will start to stretch out (Fig. 18b).

Let us return now to the shock waves produced by an explosion.

Regardless of the medium and the type of explosion, there is always in a shock wave a region of strongly compressed gas or liquid, moving in space with supersonic velocity.

When the shock wave approaches some point of space, the pressure, density, and other hydrodynamic elements at this point increase abruptly. This is followed by a gradual change of these quantities, and after some time interval the pressure and density at the given point of space will become smaller than the same parameters in the unper-

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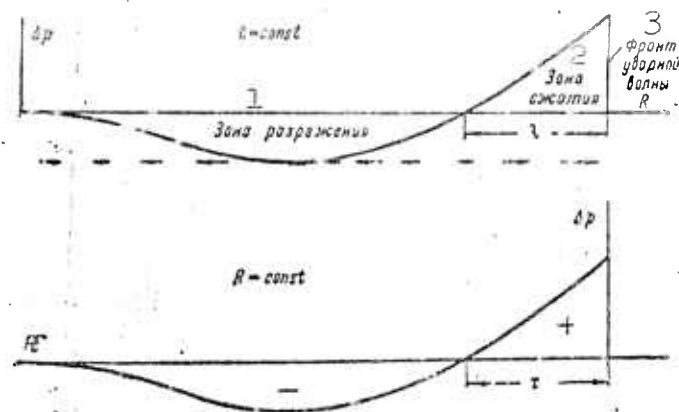
Fig. 19. Variation of pressure behind
the front of a shock wave. 1) Rarefac-
tion zone; 2) compression zone; 3)
shock-wave front.

turbed medium.

The velocity of motion of the particles decreases gradually,
changing its direction subsequently.

The qualitative character of the time and space variation of pres-
sure in a shock wave can be represented by the patterns shown in Fig.
19.

As already mentioned earlier, the forward boundary of the shock
wave is called the front.

The shock-wave pattern includes regions of positive and negative
excess pressures (compression zone and rarefaction zone). The spatial
extent of the compression zone is customarily called the length of the
shock wave, λ. It is considerably shorter than the rarefaction zone.

The time of action of the positive excess pressure is frequently
denoted by τ or t_+ and is called the period of the wave.

The area bounded by the pressure pattern and by the abscissa axis
in the compression zone is called the total impulse of the pressures
at the given point.

In a more general formulation, the impulse of the shock wave refers
to the integral with a variable upper limit.

$$J = \int_0^t p(t)\, dt. \qquad (1.178)$$

In many problems it is also of interest to estimate the energy flux density in the shock wave, which along with the pressure on the front and with the impulse can serve as one criterion for its destructive action. This is defined as the energy carried through a unit area in a specified interval of shock-wave action time.

The energy flux density is defined by the expression

$$E = \int_0^t \rho \vec{v} \Delta \left(\frac{u}{A} + \frac{v^2}{2} + \frac{p}{\rho} \right) dt, \qquad (1.179)$$

where $\rho(u/A + v^2/2)$ is the total amount of energy per unit volume of the liquid. The expression $p\vec{v}$ characterizes the work performed by the pressure forces in the wave in a unit time. The symbol Δ indicates that we are considering the energy increment compared with the initial state of the liquid.

Approximate relations for the energy flux density will be given later on.

Returning to the description of the qualitative picture of shock-wave propagation, we note that as the distance from the center of the explosion increases, the pressure on the wave front gradually decreases and the length of the wave increases somewhat.

At the limit, with extreme distances, the shock wave goes over into a sound wave.

It has been calculated that the thickness of the shock-wave front is a quantity on the order of the mean free path of the molecule (10^{-5} to 10^{-6} cm).* Because of this, the concept of the shock-wave front as a mathematical discontinuity surface is fully justified.

In spite of such a very small thickness of the shock-wave front, it is precisely in this layer that the main dissipative processes con-

nected with the change in the entropy of the system occur.

Let us consider first the case of an explosion in an unbounded ideal incompressible medium. The velocity of propagation of the perturbations in such a medium is infinitely large. The shock wave goes instantaneously to infinity. The gas bubble begins to expand. At a fixed instant, the pressure in the bubble becomes equal to the pressure in the surrounding medium. Owing to the inertia of the medium, the expansion of the gas bubble will continue. Finally, an instant sets in when under the influence of the hydrostatic pressure forces the liquid begins to move in the opposite direction and the gas bubble begins to compress. These periodic pulsations, of perfectly identical intensity, will be executed an infinite number of times.

In a compressible unbounded medium, one observes pulsations of the gas bubble, the only difference being that the energy of each succeeding pulsation will differ from the preceding one by the amount of the dissipated energy, corresponding to the change of entropy. One can present the following data, which characterize the dissipation of energy during the process of gas-bubble pulsation in the case of an underwater explosion (Table 1).

TABLE 1

1 Характерные моменты, движения газового пузыря	2 Энергия в процентах от начальной	3 Потери энергии за пульсацию, %
4 Начало 1-й пульсации	100	
2-й	41	59
3-й	21	20
4-й	14	7

1) Characteristic instants of motion of the gas bubble; 2) energy as percentage of the initial energy; 3) energy losses after the pulsation, %; 4) start of first pulsation.

These quantities characterize the energy of the shock wave formed

at the start of each pulsation.

It must be borne in mind, however, that the damaging action of the shock wave is determined only indirectly from the foregoing data.

The point is that when the wave acts on some structure or another, we deal as a rule with an irreversible process, as a result of which the decisive factor is not the total cycle of liquid motion, but only a definite part of the compression zone.

In conclusion it must be noted that both from the mathematical and the physical points of view the shock wave is a concept of very general character.

Therefore the main laws governing the propagation of shock waves and the interaction of the waves with the boundary surface can be investigated regardless of the physical process that gave rise to these waves.

§9. CALCULATION OF HYDRODYNAMIC ELEMENTS ON THE FRONT OF AN AERIAL SHOCK WAVE

The conditions of dynamic compatibility, established on the basis of the general laws of physics, enable us to define uniquely all the hydrodynamic elements on the front of a shock wave, provided we know one of them.

Earlier, in the assumption of propagation of strong-discontinuity surfaces in an unperturbed medium, we have derived the following relations for an ideal gas:

$$v_\phi = \frac{\rho_\phi - \rho_0}{\rho_\phi} N; \qquad (1.56)$$

$$N^2 = \frac{\rho_\phi}{\rho_0} \frac{p_\phi - p_0}{\rho_\phi - \rho_0}; \qquad (1.56a)$$

$$v_\phi = \sqrt{(p_\phi - p_0)\left(\frac{1}{\rho_0} - \frac{1}{\rho_\phi}\right)}; \qquad (1.58)$$

$$p_\phi - p_0 = N v_\phi \rho_0; \qquad (1.57)$$

$$\frac{\rho_\phi}{\rho_0} = \frac{(k+1)p_\phi - (k-1)p_0}{(k+1)p_0 - (k-1)p_\phi}. \qquad (1.45)$$

The temperature on the front can be readily calculated from the equation of state of an ideal gas.

Since $T_f = p_f/g\rho R$ and $T_0 = p_0/g\rho_0 R$, we have

$$\frac{T_\phi}{T_0} = \frac{p_\phi}{p_0} \frac{\rho_0}{\rho_\phi} \qquad (1.180)$$

Using Relations (1.45), (1.56), (1.58), and (1.57), let us calculate the pressures, densities, temperatures, and velocities on the shock-wave front as functions of ρ_f/ρ_0 and p_f/p_0.

In the first case we obtain after simple transformations

$$\frac{p_\phi}{p_0} = \frac{\frac{k+1}{k-1}\frac{\rho_\phi}{\rho_0} - 1}{\frac{k+1}{k-1} - \frac{\rho_\phi}{\rho_0}}; \qquad (1.181)$$

$$\frac{T_\phi}{T_0} = \frac{\frac{k+1}{k-1} - \frac{\rho_0}{\rho_\phi}}{\frac{k+1}{k-1} - \frac{\rho_\phi}{\rho_0}} \qquad (1.182)$$

For the velocity of sound we previously had $a = \sqrt{kp/\rho}$ or $a = \sqrt{kgRT}$. Consequently,

$$\frac{a_\phi}{a_0} = \sqrt{\frac{T_\phi}{T_0}}; \qquad (1.183)$$

$$N = a_0 \sqrt{\frac{\frac{2}{k-1}\frac{\rho_\phi}{\rho_0}}{\frac{k+1}{k-1} - \frac{\rho_\phi}{\rho_0}}}; \qquad (1.184)$$

$$v = a_0 \left(\frac{\rho_\phi}{\rho_0} - 1\right) \sqrt{\frac{\frac{2}{k-1}\frac{\rho_0}{\rho_\phi}}{\frac{k+1}{k-1} - \frac{\rho_\phi}{\rho_0}}} \qquad (1.185)$$

In the second case

$$\frac{\rho_\phi}{\rho_0} = \frac{\frac{k+1}{k-1}\frac{p_\phi}{p_0} + 1}{\frac{k+1}{k-1} + \frac{p_\phi}{p_0}}; \qquad (1.186)$$

$$\frac{T_\phi}{T_0} = \frac{\frac{k+1}{k-1} + \frac{p_\phi}{p_0}}{\frac{k+1}{k-1}\frac{p_\phi}{p_0} + 1}\frac{p_\phi}{p_0}; \qquad (1.187)$$

$$N = a_0 \sqrt{\frac{k+1}{2k}\frac{p_\phi}{p_0} + \frac{k-1}{2k}}; \qquad (1.188)$$

$$v_\phi = \frac{a_0\left(\frac{p_\phi}{p_0} - 1\right)}{k\sqrt{\frac{k+1}{2k}\frac{p_\phi}{p_0} + \frac{k-1}{2k}}}. \qquad (1.189)$$

From Formulas (1.186)-(1.189) it follows, in particular, that if

any of the hydrodynamic elements on the front of the shock wave tends to its value in the unperturbed liquid, then all the remaining elements will also tend to their values in the unperturbed medium.*

In many problems it is necessary to know besides the cited parameters of the wave front, also the value of the velocity head $\rho v^2/2$. On the basis of (1.186) and (1.189) we obtain for it, after elementary transformations

$$\Delta p_{ck} = \frac{2{,}5\,(p_\phi - p_0)^2}{p_\phi + 6p_0} \qquad (1.192)$$

For the parameters characterizing the unperturbed medium one usually assumes the values for the international standard atmosphere:

$$p_0 = 1.0332 \text{ kg/cm}^2 \ (760 \text{ mm Hg});$$
$$\rho_0 = 0.125 \text{ kg-sec}^2/\text{m}^4;$$
$$a_0 = 340 \text{ m/sec}.$$

After substituting these values, the formulas characterizing the propagation of an aerial shock wave at the earth's surface can be written in the form

$$N = 340\sqrt{1 + 0{,}83\,\Delta p_\phi} \text{ m/sec}, \qquad (1.193)$$

$$v_\phi = \frac{235\Delta p_\phi}{\sqrt{1 + 0{,}83\Delta p_\phi}} \text{ m/sec}, \qquad (1.194)$$

$$\rho_\phi = 0{,}125\frac{6\Delta p_\phi + 7{,}2}{\Delta p_\phi + 7{,}2} \text{ kg} \cdot \text{sec}^2/\text{m}^4; \qquad (1.195)$$

$$T_\phi = 288\frac{(1 + \Delta p_\phi)\,(\Delta p_\phi + 7{,}2)}{6\Delta p_\phi + 7{,}2} \text{ °K}; \qquad (1.196)$$

$$a_\phi = 20{,}1\sqrt{T_\phi} \text{ m/sec}. \qquad (1.197)$$

In Table 2 are given the values of the hydrodynamic elements on the front of the wave as functions of the pressure, calculated from Formulas (1.186)-(1.192).** The plots of N, $\underline{v}$, ρ, T, a_0, and Δp_{sk} as functions of the pressure are shown in Figs. 20 and 21.

They characterize the air shock wave with high accuracy in the range of front pressures to 50 atm.

What is striking is the fact that starting with pressures Δp on

TABLE 2

$\Delta p = p - p_{0}$, kg/cm² [1]	N, m/sec [2]	v, m/sec	ρ, kg-sec²/m⁴ [3]	T, °K	a, m/sec	Δp_{ck}, kg/cm²	Δs, cal/g-deg [4]
0,00	340	0	0,1250	288	340	0	—
0,01	341	2,34	0,1258	289	341	$3,31\cdot10^{-4}$	—
0,10	354	22,6	0,1335	296	345	$3,42\cdot10^{-3}$	$0,628\cdot10^{-4}$
0,20	367	43,6	0,142	303	349	$1,34\cdot10^{-2}$	$0,320\cdot10^{-4}$
0,30	380	63,2	0,150	310	353	$2,99\cdot10^{-2}$	$0,106\cdot10^{-3}$
0,40	392	82,0	0,158	316	356	$5,16\cdot10^{-2}$	$0,180\cdot10^{-3}$
0,50	404	99,2	0,165	323	360	$7,58\cdot10^{-2}$	$0,255\cdot10^{-3}$
0,60	416	115	0,173	329	364	0,109	$0,340\cdot10^{-}$
0,80	439	146	0,188	341	371	0,196	$0,523\cdot10^{-}$
1,00	460	174	0,201	353	377	0,299	$0,740\cdot10^{-3}$
1,20	480	200	0,214	364	383	0,417	$1,00\cdot10^{-3}$
1,40	500	224	0,227	375	389	0,558	$1,40\cdot10^{-3}$
1,60	519	247	0,239	385	394	0,703	$2,36\cdot10^{-2}$
1,80	537	268	0,250	395	400	0,886	$4,07\cdot10^{-3}$
2,00	555	287	0,261	405	404	1,06	$6,08\cdot10^{-3}$
2,25	576	313	0,274	418	411	1,3?	$6,20\cdot10^{-3}$
2,50	596	336	0,286	431	417	1,59	$9,00\cdot10^{-3}$
2,75	616	358	0,299	442	423	1,86	0,0100
3,00	635	378	0,309	455	428	2,19	0,0162
3,50	672	417	0,329	480	439	2,85	0,0180
4,00	707	453	0,349	503	450	3,59	0,0246
4,50	740	486	0,366	527	461	4,27	0,0280
5,00	772	518	0,381	552	471	5,07	0,0320
5,50	802	548	0,395	575	481	5,93	0,0370
6,00	832	576	0,408	600	491	6,73	0,0431
7 00	888	630	0,432	649	511	8,58	0,0510
8,00	940	680	0,453	697	530	10,5	0,0602
9,00	990	727	0,472	742	546	12,5	0,0680
10,0	1 040	772	0,489	787	562	14,6	0,0769
20,0	1 430	1 120	0,585	1 250	710	36,6	0,149
30,0	1 730	1 380	0,629	1 720	833	60,4	0,192
40,0	1 990	1 610	0,655	2 180	936	84,6	0,230
50,0	2 220	1 800	0,672	2 650	1 030	109,5	0,263
70,0	2 620	2 180	0,731	3 340	—	—	0,310
100,0	3 110	2 580	0,806	4 160	—	—	0,370
150,0	3 810	3 230	0,937	5 060	—	—	0,435
200,0	4 420	3 740	1,088	5 680	—	—	0,500
250,0	4 900	4 280	1,230	6 230	—	—	0,570
300,0	5 380	4 760	1,315	6 740	—	—	0,630
350,0	5 840	5 160	1,360	7 230	—	—	0,700
400,0	6 180	5 570	1,375	7 750	—	—	0,755
500,0	6 940	6 330	—	8 700	—	—	0,885
600,0	7 580	6 940	—	9 650	—	—	1,000
800,0	8 740	7 960	—	11 250	—	—	1,200
1000,0	9 780	8 840	—	13 600	—	—	1,400
1200,0	10 600	9 600	—	15 800	—	—	1,570
1400,0	11 400	10 350	—	18 200	—	—	1,700
1600,0	12 200	10 950	—	21 600	—	—	1,845

1) kg/cm²; 2) m/sec; 3) kg-sec²/m⁴; 4) cal/g-deg.

the front exceeding 4 atm, the velocity of the air particles becomes larger than the local velocity of sound.

The propagation of a strong aerial shock wave causes the formation of a supersonic stream. The increase in the pressure on the front

Fig. 20. Hydrodynamic elements on the front of an aerial shock wave as functions of the pressure ($\Delta p_{\phi} < 2.0$ kg/cm^2).

Fig. 21. Hydrodynamic elements on the front of an aerial shock wave as functions of the pressure ($\Delta p_f < 50$ kg/cm^2).

is accompanied by a rather appreciable increase in the temperature. Thus, for $\Delta p \simeq 40$ atm the temperature reaches about 2000°C.

In the region of very high pressures, the thermodynamic properties of air already differ greatly from the properties of an ideal gas

Such a difference is determined principally by the dissociation and ionization of the medium, processes that occur quite intensely at high pressures and temperatures.

A theoretical scheme for taking into account the ionization and dissociation in the propagation of shock waves in air is given in the book "Physics of Explosions" of F.A. Baum, K.P. Stanyukovich, and B.I. Shekhter.

The same book presents an analysis of the corresponding calculations made by Burkhardt. It is shown, in particular, that the rate of displacement of the shock wave differs insignificantly from the calculated values obtained using the ideal-gas scheme. Appreciable differences are observed, to the contrary, in the estimates of the densities and temperatures. For example, when $p_f/p_0 = 3000$ the density calculated by the classical method is 2 or 3 times higher than the results of Burkhardt's calculations.

To allow an approximate estimate of the hydrodynamic elements on the front of the wave in the near zone of the explosion, Table 2 has been supplemented by Burkhardt's computational data ($\Delta p > 50$ atm).

Example 1. Calculate the velocity of front propagation of an aerial shock wave, if it is known that the excess pressure on the front amounts to $\Delta p_f = 120$ atm.

Solution. The propagation velocity of the shock wave θ is defined as the velocity with which the shock wave moves relative to the moving gas particles: $\theta = N - v_f$.

Consequently, in order to calculate the propagation velocity θ, it is necessary to know the velocity of the shock wave and the velocity v_f of the particles behind the front. In accordance with (1.193) we have $N = 340\sqrt{1 + 0.83\ \Delta p_f} = 340\sqrt{1 + 0.83 \cdot 1.2} = 340 \cdot 1.41 = 480$ m/sec. On the basis of (1.193) and (1.194) we have

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$$v_\phi = \frac{8 \cdot 10^4 \cdot \Delta p_\phi}{N} = \frac{8 \cdot 10^4 \cdot 1,2}{480} = 200 \text{ m/sec.}$$

Consequently, $\theta = 480 - 200 = 280$ m/sec.

Example 2. Calculate the pressure Δp_f, the air density ρ_f, the particle displacement velocity v_f, the velocity of sound a_f, and the temperature on the front of the wave, if it is established that the shock-wave displacement velocity at a certain point amounts to N = 555 m/sec.

Solution. We first determine from Formula (1.193) the value of the excess pressure on the front of the shock wave:

$$N = 340 \sqrt{1 + 0,83 \Delta p_\phi};$$
$$555^2 = 340^2 (1 + 0,83 \Delta p_\phi),$$
$$\Delta p_\phi = \frac{1}{0,83} \left[\left(\frac{555}{340} \right)^2 - 1 \right] = 2,00 \text{ kg/cm}^2. \tag{1.193}$$

Further calculations are based on Formulas (1.194)-(1.197):

$$v_\phi = \frac{8 \cdot 10^4 \Delta p_\phi}{N} = \frac{8 \cdot 10^4 \cdot 2,00}{555} = 287 \text{ m/sec};$$

$$\rho_\phi = 0,125 \frac{6 \Delta p_\phi + 7,2}{\Delta p_\phi + 7,2} = 0,125 \frac{6 \cdot 2,0 + 7,2}{2,0 + 7,2} = 0,261 \text{ kg-sec}^2/\text{m}^4;$$

$$T_\phi = 288 \frac{(1 + \Delta p_\phi)(\Delta p_\phi + 7,2)}{6 \Delta p_\phi + 7,2} = 288 \frac{(1 + 2,00)(2,00 + 7,2)}{6 \cdot 2,0 + 7,2} = 405 \text{ °K};$$

$$a_\phi = 20,1 \sqrt{T_\phi} = 20,1 \sqrt{405} = 404 \text{ m/sec.}$$

§10. CALCULATION OF THE HYDRODYNAMIC ELEMENTS ON THE FRONT OF AN UNDER-WATER SHOCK WAVE

In perfect analogy with the procedure used for the ideal gas, using the equations of dynamic compatibility and the equation of state of water, we can establish a unique connection between the hydrodynamic elements on the front of an underwater shock wave.

A simple equation of state of water is available at present for only part of the possible range of parameters.* The equation is written differently for the region of isentropic flow and for motion with variable entropy. Naturally, the conditions of dynamic compatibility that result from the equation of state are also different, depending on the range of variation of temperatures and pressures to be investi-

gated.

Leaving out the rather cumbersome derivations, we present only the final results.

For the region of high pressures and temperatures ($p > 25 \cdot 10^3$ atm)* we have

$$p_\phi - p_0 = d\,(\rho_\phi^k - \rho_0^k)\ [\text{kg/cm}^2];\qquad (1.198)$$

$$N^2 = \frac{d\,(\rho_\phi^k - \rho_0^k)}{\rho_0\left(1 - \frac{\rho_0}{\rho_\phi}\right)}\ \text{m}^2/\text{sec}^2;\qquad (1.199)$$

$$v = \frac{d\,(\rho_\phi^k - \rho_0^k)}{\rho_0}\left(1 - \frac{\rho_0}{\rho_\phi}\right).\qquad (1.200)$$

where $d = 4250$; $k = 6.29$.

For isentropic motion $p < 25 \cdot 10^3$ we have

$$\frac{p_\phi + B}{\rho_\phi^n} = \frac{p_0 + B}{\rho_0^n};\qquad (1.201)$$

$$N^2 = \frac{p_\phi}{\rho_0\left(1 - \frac{\rho_0}{\rho_\phi}\right)};\qquad (1.202)$$

$$v^2 = \frac{p_\phi}{\rho_0}\left(1 - \frac{\rho_0}{\rho_\phi}\right);\qquad (1.203)$$

$$a^2 = \frac{n\,(p + B)}{\rho}.\qquad (1.204)$$

Figure 22 shows the dynamic adiabatic curves corresponding to Eqs. (1.198) and (1.201).

The point of contiguity has coordinates $p = 27,580$ kg/cm^2 and $\rho = 1.364$ g/cm^3.

Passing through the same point is a Poisson adiabatic curve $[(p + C)/\rho^*] = (\rho/\rho^*)^\kappa$ with exponent $\kappa = 5.5$.

In the study of the propagation of a shock wave with a pressure on the front not exceeding 1000 atm, the conditions of dynamic compatibility can be linearized.

Neglecting the value of p_0 compared with B, the isentropic condition can be written in the form

$$\frac{p_\phi + B}{\rho_\phi^n} = \frac{B}{\rho_0^n}$$

or

$$p = B\left[\left(\frac{\rho_\Phi}{\rho_0}\right)^n - 1\right].$$

The velocity of sound in the unperturbed liquid will therefore be

$$a_0 = \sqrt{\frac{nB}{\rho_0}}.$$

Fig. 22. Comparison of dynamic adiabatic curves for water as given by different formulas. Symbols:

1) Dynamic adiabatic curve.

The velocity of sound on the wave front can obviously be calculated with the aid of the relation

$$a_\Phi = \sqrt{\frac{dp_\Phi}{d\rho_\Phi}} = \sqrt{\frac{Bn}{\rho_0}\left(\frac{\rho}{\rho_0}\right)^{\frac{n-1}{n}}} = a_0\sqrt{\left(\frac{\rho}{\rho_0}\right)^{\frac{n-1}{n}}},$$

but

$$\sqrt{\left(\frac{\rho}{\rho_0}\right)^{\frac{n-1}{n}}} = \left(\frac{p}{B}+1\right)^{\frac{n-1}{n}} \simeq 1 - \frac{n-1}{2n}\frac{p}{B}.$$

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Thus

$$a_\phi = a_0 \left[1 + \frac{n-1}{2Bn} p_\phi \right]$$

Neglecting p_0 compared with p, we can rewrite (1.55a) and (1.57) in the form

$$N^2 = \frac{\rho_\phi}{\rho_0 \left(1 - \frac{\rho_0}{\rho_\phi} \right)},$$

$$p_\phi = \rho_0 N v_\phi,$$

hence

$$v_\phi^2 = \frac{p_\phi}{\rho_0} \left(1 - \frac{\rho_0}{\rho_\phi} \right).$$

Since

$$\frac{\rho_0}{\rho} = \frac{1}{\left(1 + \frac{p}{B} \right)^{\frac{1}{n}}}, \quad v^2 = \frac{p}{\rho_0} \left[1 - \frac{1}{\left(1 + \frac{p_\phi}{B} \right)^{\frac{1}{n}}} \right] \cong \frac{Bn}{\rho_0} \frac{p_\phi^2}{Bn(Bn + p_\phi)}$$

and

$$p_\phi \ll Bn,$$

we have

$$v^2 = a_0^2 \frac{p_\phi^2}{B^2 n^2}$$

or

$$v = a_0 \frac{p_\phi}{Bn}.$$

Carrying out similar derivations for the quantity N, we obtain

$$N^2 = \frac{p_\phi}{\rho_0} \left(1 + \frac{p_\phi}{B} \right)^{\frac{1}{n}} \frac{1}{\left(1 + \frac{p_\phi}{B} \right)^{\frac{1}{n}} - 1} =$$

$$= \frac{p_\phi}{\rho_0} \left(1 + \frac{p_\phi}{Bn} \right) \frac{1}{\frac{p_\phi}{Bn} + \frac{1}{2n} \left(\frac{1}{n} - 1 \right) \frac{p_\phi^2}{B^2}} =$$

$$= a_0^2 \left(1 + \frac{1+n}{2Bn} p_\phi \right).$$

or

$$N = a_0 \left(1 + \frac{1+n}{4Bn} p_\phi \right).$$

Thus, the linearized conditions of dynamic compatibility for pres-

sures $p_f < 1000$ atm can be written in the form

$$N = a_0 \left(1 + \frac{i+n}{4Bn} p_\phi\right);$$
(1.205)

$$a_\phi = a_0 \left(1 + \frac{n-1}{2Bn} p_\phi\right);$$
(1.206)

$$v_\phi = a_0 \frac{p_\phi}{Bn};$$
(1.207)

$$T_\phi \simeq T_0.$$

In parallel with the theoretical investigations, many researchers have made attempts to obtain the conditions of dynamic compatibility of water by experiment.

The most complete and conclusive results were obtained by R. Schall. With the aid of high-speed photography he was able to establish the displacement velocity of the shock wave at different instants of time. Pulsed x-ray photography has made it possible to determine the density on the front of the shock wave from the absorption coefficients of the rays.

Since

$$\rho_0 N = \rho_\phi (N - v_\phi)$$

and

$$p_\phi = \rho_0 v_\phi N,$$
$$p_\phi = \rho_0 N^2 \left(1 - \frac{\rho_0}{\rho_\phi}\right),$$

it is easy to obtain, by establishing experimentally the connection between N and ρ, empirical relations replacing the dynamic adiabatic curve for water.

Figure 23 shows the dependence $N/a_0 = f(v_f/a_0)$ as given by the experimental data of R. Schall.

It must be noted that with sufficiently good approximation, the experimental and the calculated data fall on a line passing through the origin for pressures up to approximately $20 \cdot 10^3$ atm, corresponding to $v/a_0 \simeq 0.6$.

The equation of this line is

$$\frac{N}{a_0} = 1 + m \frac{v}{a_0},$$ (1.208)

where $m \simeq 2.0$.

Equation (1.208) can be regarded as a third dynamic-compatibility condition, established by experiment. We previously had

$$\rho_0 N = \rho_\Phi (N - v_\Phi);$$ (1.56)

$$\rho_0 N v_\Phi = p_\Phi - p_0.$$ (1.57)

By elementary transformations we can readily obtain from (1.56), (1.57), and (1.208)*

$$v = a_0 \frac{1 - \frac{\rho_0}{\rho_\Phi}}{1 - m\left(1 - \frac{\rho_0}{\rho_\Phi}\right)};$$ (1.209)

$$N = a_0 \frac{1}{1 - m\left(1 - \frac{\rho_0}{\rho_\Phi}\right)};$$ (1.210)

$$p_\Phi = \rho_0 a_0^2 \frac{1 - \frac{\rho_0}{\rho_\Phi}}{\left[1 - m\left(1 - \frac{\rho_0}{\rho_\Phi}\right)\right]^2}.$$ (1.211)

If it is necessary to represent the hydrodynamic quantities as functions of the displacement velocity of the shock wave N, a simultaneous solution of Eqs. (1.208), (1.56), and (1.57) yields the relations

$$v_\Phi = \frac{1}{m}(N - a_0);$$ (1.212)

$$\rho_\Phi = \frac{1}{1 - \frac{1}{m}\left(1 - \frac{a_0}{N}\right)};$$ (1.213)

$$p_\Phi = \frac{1}{m}\rho_0 N(N - a_0).$$ (1.214)

By comparison of Figs. 22 and 23 we can readily verify that Formulas (1.209)-(1.214) are equivalent to the conditions of dynamic compatibility (1.201)-(1.203).

The use of formulas of one type or another depends on the convenience in the solution of one problem or another.
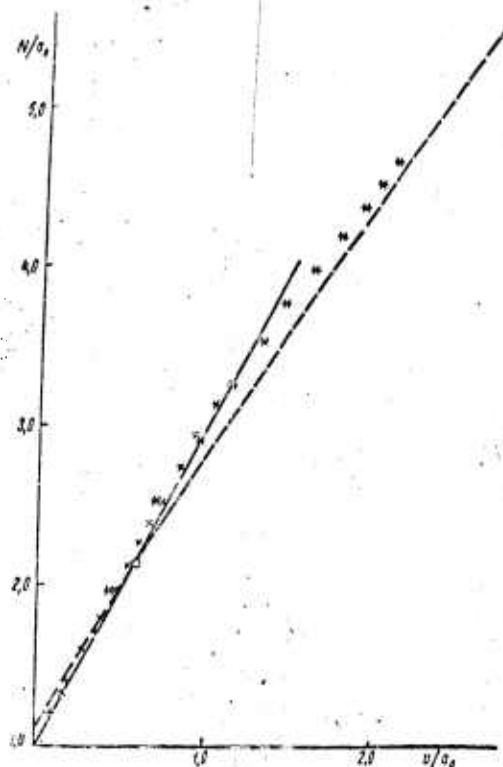
Fig. 23. Dependence $N/a_0 = f(v/a_0)$ as given by experimental and theoretical data.

Notation:

*) Schall's experiment;

+) calculation by Zel'dovich and Leypunskiy;

×) calculations after R. Koul;

——) from the adiabatic curve $(p+B)/\rho^n = (p_0+B)/\rho_0^n$;

- - -) from the adiabatic curve $p_f - p_0 = d(\rho_f^k - \rho_0^k)$;

o) $N/a_0 = 2.14$; $v/a_0 = 0.586$; $p = 27,580$ kg/cm^2.

For the sake of clarity the calculation of the hydrodynamic elements on the front of an underwater shock wave is summarized in Table 3 and graphically represented on Figs. 24 and 25.

Unlike air, no supersonic motion of the liquid occurs even at very high pressures on the wave front, and the temperature rise is relatively slow.

TABLE 3

1 кг/см² p	2 м/сек N	v м/сек	3 г/см³ ρ	a м/сек	Δ, T°C	4 кал/град Δs
0	1 460	0	1,000	1 460	0	0
200	1 490	13	1,013	1 500	2,0	0,348·10⁻³
400	1 510	26	1,021	1 540	2,4	0,199·10⁻²
600	1 540	40	1,032	1 580	2,6	0,498·10⁻⁴
800	1 560	58	1,040	1 620	3,0	0,933·10⁻⁴
1 000	1 590	67	1,041	1 660	3,4	0,154·10⁻³
1 200	1 615	80	1,053	1 700	3,8	0,229·10⁻³
1 400	1 640	93	1,058	1 740	4,0	0,293·10⁻³
1 600	1 670	106	1,065	1 780	4,4	0,416·10⁻²
1 800	1 685	120	1,070	1 820	4,8	0,522·10⁻³
2 000	1 720	133	1,075	1 860	5,8	0,648·10⁻³
2 200	1 745	146	1,080	1 900	6,0	0,790·10⁻³
2 400	1 775	160	1,085	1 940	7,0	0,948·10⁻³
2 600	1 800	173	1,090	1 980	8,0	0,00113
2 800	1 825	185	1,095	2 020	8,4	0,00134
3 000	1 850	200	1,100	2 060	8,8	0,00157
4 000	1 940	240	1,120	2 160	14,0	0,00276
5 000	2 040	280	1,140	2 240	18,0	0,0045
6 000	2 100	320	1,160	2 300	22,0	0,0070
7 000	2 190	360	1,175	2 420	24,0	0,0100
8 000	2 240	400	1,200	2 500	30,0	0,0125
9 000	2 300	420	1,210	2 600	32,0	0,0160
10 000	2 400	450	1,220	2 660	35,5	0,0190
15 000	2 660	580	1,275	2 960	51,0	0,0380
20 000	2 840	680	1,325	3 200	68,0	0,0600
25 000	3 060	800	1,360	3 470	85,0	0,0800
30 000	3 260	930	1,400	3 720	105,0	0,102
40 000	3 600	1 100	1,450	4 010	136	0,152
50 000	3 900	1 280	1,500	4 415	184	0,206
60 000	4 140	1 430	1,545	4 740	214	0,268
70 000	4 400	1 540	1,580	4 940	260	0,320
80 000	4 600	1 680	1,615	5 162	300	0,378
90 000	4 800	1 800	1,640	5 400	340	0,425
100 000	5 000	1 940	1,665	5 600	400	0,488
200 000	6 460	3 000	1,850	7 100	870	0,944
300 000	7 800	3 800	1,970	8 160	1390	1,283

1) kg/cm^2; 2) m/sec; 3) g/cm^3; 4) cal/g-deg.

Example 1. Find the wave-displacement velocity N, the local velocity of sound a_f, the velocity of the particles behind the front v_f, and the density ρ_f, if the pressure on the front of an underwater shock wave amounts to 1000 kg/cm^2.

Solution. For the specified value of the pressure, the calculation can be carried out using the linearized dynamic-compatibility conditions (1.205)-(1.207).

We have:

$$N = a_0 \left(1 + \frac{n+1}{4Bn} p_\phi\right) = 1460 \left(1 + \frac{7{,}15+1}{4 \cdot 3045 \cdot 7{,}15} \cdot 1000\right) = 1590 \ \text{m/sec};$$

$$a_\phi = a_0 \left(1 + \frac{n-1}{2Bn} p_\phi\right) = 1460 \left(1 = \frac{(7{,}15-1) \cdot 1000}{2 \cdot 3045 \cdot 7{,}15}\right) = 1600 \ \text{m/sec};$$

$$v_\phi = a_0 \frac{p_\phi}{Bn} = 1460 \frac{1000}{3045 \cdot 7{,}15} = 67{,}0 \ \text{m/sec};$$

$$\left(\frac{\rho_\phi}{\rho_0}\right)^n = \frac{p_\phi + B}{p_0 + B} = \frac{1000 + 3045}{3046} = 1{,}328;$$

$$\lg \frac{\rho_\phi}{\rho_0} = \frac{\ln 1{,}328}{7{,}15} = \frac{0{,}123}{7{,}15} = 0{,}0172;$$

$$\rho_\phi = 1{,}04 \ \text{g/cm}^3.$$
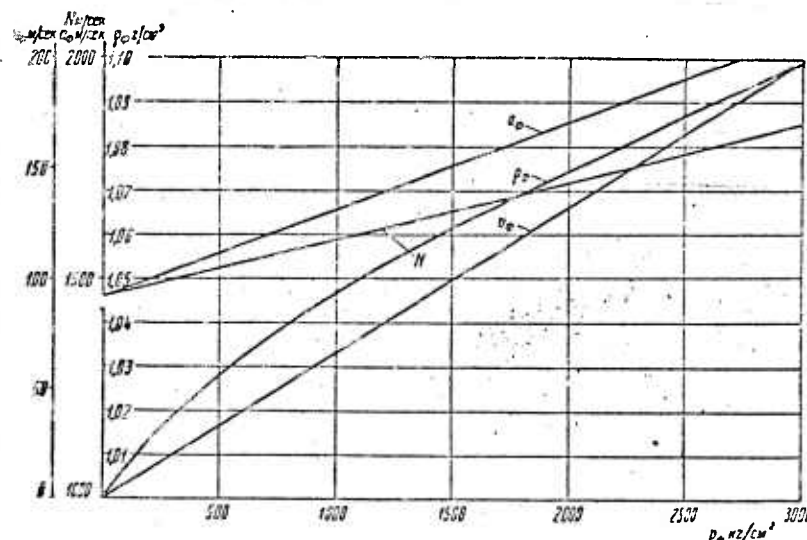
Fig. 24. Hydrodynamic elements on the front of an underwater shock wave as a function of the pressure ($p_f < 3000 \ \text{kg/cm}^2$).

Example 2. Calculate the velocity of sound a_f behind the front of an underwater shock wave, if the pressure on the front amounts to 237 thousand atm (the coefficient κ is in this case equal to 4.63).

Solution. From the dynamic-compatibility condition (1.198) we calculate the value of the density on the wave front:

$$p_\phi - p_0 = d\left(\rho_\phi^\kappa - \rho_0^\kappa\right);$$
$$237\,000 = 4250\left(\rho_\phi^{6{,}29} - 1\right);$$
$$\rho_\phi^{6{,}29} = 56{,}8;$$
$$\rho_\phi = 1{,}90 \ \text{g/cm}^3.$$

Following the jump increase in the pressure and density as a result of the passage of the wave through the given point, the water be-
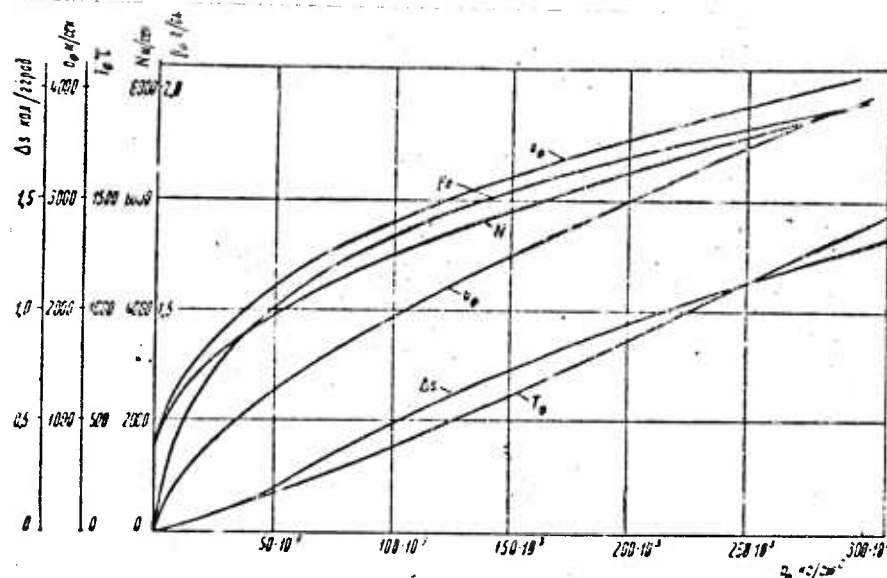
Fig. 25. Hydrodynamic elements on the front of
an underwater shock wave as a function of the
pressure ($p_f < 300 \cdot 10^3$ kg/cm^2).

gins to expand adiabatically.

The adiabaticity condition is determined by the relation (1.21)

$$\frac{p + c}{\rho^{\kappa}} = \left(\frac{\rho}{\rho^*}\right)^{\kappa}.$$

Differentiating this equation, we obtain

$$a^2 = \frac{dp}{d\rho} = \frac{\kappa}{\rho}(p + c).$$

Expressing the pressure in dynes per square centimeter (1 kg =
= $9.81 \cdot 10^5$ g-cm/sec^2), we obtain

$$a = \sqrt{\frac{4 \cdot 63}{1 \cdot 90}(237\,000 + 5\,400)\,9.81 \cdot 10^6} = 761\,200 \text{ м/сек} = 7612 \text{ m/sec.} \quad \text{[sic]}$$

§11. NORMAL REFLECTION OF A PLANE SHOCK WAVE FROM AN ABSOLUTELY RIGID WALL

Assume that a plane shock wave is normally incident on an infin-
ite absolutely rigid wall (Fig. 26). The shock-wave displacement veloc-
ity N will be assumed directed normal to the wall. The parameters of
the unperturbed medium are denoted p_0, ρ_0, T_0, and $v_0 = 0$, while the
parameters of the medium behind the front of the direct wave are de-

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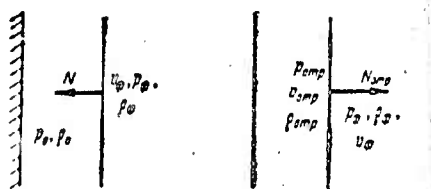
Fig. 26. Reflection of a plane shock wave from an unbounded absolutely rigid wall.

noted by p_f, ρ_f, T_f, and v_f.

At the instant when the shock wave encounters the wall, a reflected wave is produced, propagating in the opposite direction with velocity N_{otr}. Its parameters are denoted by p_{otr}, ρ_{otr}, T_{otr}, and v_{otr}.

The normal component of the particle velocity on the wall is equal to zero in accordance with the conditions of the problem.

According to Formula (1.55), we have for the direct wave

$$v_\phi = \sqrt{(p_\phi - p_0)\left(\frac{1}{\rho_0} - \frac{1}{\rho_\phi}\right)}.$$

In accordance with the same general relationship we have

$$v_{отр} + v_\phi = \sqrt{(p_{отр} - p_\phi)\left(\frac{1}{\rho_\phi} - \frac{1}{\rho_{отр}}\right)}. \tag{1.215}$$

Recognizing that on the wall $v_{otr} = 0$, we obtain on the basis of (1.55) and (1.215):*

$$\sqrt{(p_{отр} - p_\phi)\left(\frac{1}{\rho_\phi} - \frac{1}{\rho_{отр}}\right)} = \sqrt{(p_\phi - p_0)\left(\frac{1}{\rho_0} - \frac{1}{\rho_\phi}\right)},$$

or

$$(p_\phi - p_0)\left(\frac{1}{\rho_0} - \frac{1}{\rho_\phi}\right) = (p_{отр} - p_\phi)\left(\frac{1}{\rho_\phi} - \frac{1}{\rho_{отр}}\right),$$

hence

$$\frac{p_{отр} - p_\phi}{p_\phi - p_0} = \frac{\frac{1}{\rho_0} - \frac{1}{\rho_\phi}}{\frac{1}{\rho_\phi} - \frac{1}{\rho_{отр}}}. \tag{1.216}$$

Equation (1.216) can be rewritten in the form

$$\cdot\frac{p_{отр} - p_0}{p_\phi - p_0} = 1 + \frac{\frac{1}{\rho_0} - \frac{1}{\rho_\phi}}{\frac{1}{\rho_\phi} - \frac{1}{\rho_{отр}}} = \frac{\frac{\rho_{отр}}{\rho_0} - 1}{\frac{\rho_{отр}}{\rho_\phi} - 1}. \tag{1.217}$$

Let the medium in which the reflection is being considered be an ideal gas. Then, according to the equation of the shock adiabatic curve

(1.186), we have

$$\frac{\rho_\Phi}{\rho_0} = \frac{\frac{k+1}{k-1}\frac{p_\Phi}{p_0} + 1}{\frac{k+1}{k-1} + \frac{p_\Phi}{p_0}};$$
(1.186)

$$\frac{\rho_{отр}}{\rho_\Phi} = \frac{\frac{k+1}{k-1}\frac{p_{отр}}{p_\Phi} + 1}{\frac{k+1}{k-1} + \frac{p_{отр}}{p_\Phi}}.$$
(1.218)

Thus, we have obtained two additional equations, with the aid of which we can eliminate ρ_{otr} and ρ_f from (1.217). Carrying out the corresponding transformations, we obtain

$$\frac{p_{отр}}{p_\Phi} = \frac{\frac{3k+1}{k-1}\frac{p_\Phi}{p_0} - 1}{\frac{k+1}{k-1} + \frac{p_\Phi}{p_0}}.$$
(1.219)
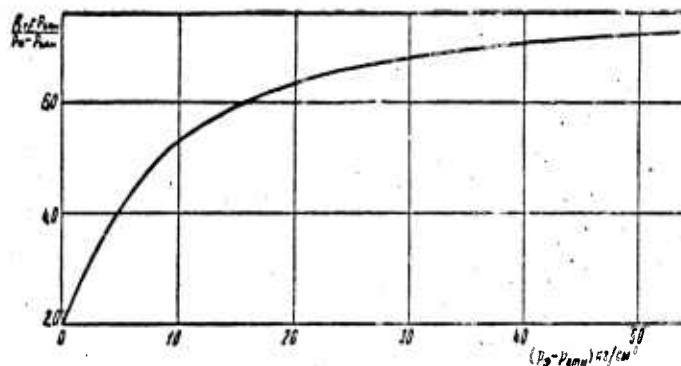
Fig. 27. Plot of the function
$(p'_{otr} - p_{atm})/(p_f - p_{atm}) = f(p_f)$ for
normal reflection of an aerial shock
wave from a partition.

Formula (1.219) is called the Izmaylov-Crussard formula. It can be reduced to the form

$$\frac{p_{отр} - p_0}{p_\Phi - p_0} = 2 + \frac{\frac{k+1}{k-1}(p_\Phi - p_0)}{(p_\Phi - p_0) + \frac{2k}{k-1}p_0},$$
(1.220)

or, what is the same,

$$\Delta p_{отр} = 2\Delta p_\Phi + \frac{6\Delta p_\Phi^2}{\Delta p_\Phi + 7p_0}.$$
(1.221)

A plot of the function $(p_{otr} - p_0)/(p_f - p_0) = f(p_f)$ is shown in

Fig. 27. It follows from Formula (1.221) that for a shock wave of low amplitude ($p_f \rightarrow p_0$) we have

$$\frac{p_{\text{отр}} - p_0}{p_\phi - p_0} = 2.$$

Thus, the excess pressure on the partition doubles compared with the excess pressure behind the front of the shock wave, in agreement with the well-known result for the reflection of acoustic waves.

In the other limiting case, when $p_f - p_0 \rightarrow \infty$, we have for $k = 1.4$

$$\frac{p_{\text{отр}} - p_0}{p_\phi - p_0} = 8.$$

Using the conditions of dynamic compatibility, it is easy to determine all the remaining hydrodynamic elements of the wave at the wall. As a result of simple calculations we obtain

$$\frac{p_{\text{отр}}}{p_\phi} = \frac{k \dfrac{p_\phi}{p_0}}{(k-1)\dfrac{p_\phi}{p_0} + 1}; \qquad (1.222)$$

$$\frac{T_{\text{отр}}}{T_\phi} = \frac{p_{\text{отр}}}{p_\phi} \frac{p_\phi}{p_{\text{отр}}}; \qquad (1.223)$$

$$N_{\text{отр}} = \left[(k-1)\frac{p_\phi}{p_0} + 1\right] \sqrt{\frac{2\dfrac{p_0}{p_0}}{(k+1)\dfrac{p_\phi}{p_0} + (k-1)}}. \qquad (1.224)$$

Comparison of the results of the calculation for an ideal gas using the formulas obtained with the results of more complicated calculations for air, with account of the dependence of its specific heat on the temperature, shows that the relationships presented can be used with accuracy sufficient for practical purposes when $p_f/p_0 < 40$.

In perfectly analogous fashion we can calculate the pressure on an absolutely rigid partition for reflection of an underwater shock wave.

Here, obviously, the relation

$$\frac{p_{\text{отр}} - p_0}{p_\phi - p_0} = \frac{\dfrac{p_{\text{отр}}}{p_0} - 1}{\dfrac{p_{\text{отр}}}{p_\phi} - 1}, \qquad (1.217)$$

established from the general conditions of dynamic compatibility, will hold true also for water.

For a unique estimate of the ratio p_{otr}/p_f it is necessary to have one more equation relating the pressure with the density.

Such an equation, generally speaking, should be the equation of the dynamic adiabatic curve.

However, as was already mentioned above, for the range of pressures up to $25 \cdot 10^3$ atm, it can be replaced successfully by the Poisson adiabatic curve equation, since the change in entropy is negligibly small.

In this case

$$\frac{p_\Phi + B}{\rho_\Phi^n} = A, \tag{1.201}$$

from which it follows that

$$\frac{\rho_{отр}}{\rho_0} = \left[\frac{p_{отр} + B}{p_0 + B}\right]^{\frac{1}{n}}; \tag{1.225}$$

$$\frac{\rho_{отр}}{\rho_\Phi} = \left[\frac{p_{отр} + B}{p_\Phi + B}\right]^{\frac{1}{n}}. \tag{1.226}$$

Simultaneous solution of Eqs. (1.217), (1.225), and (1.226) enables us to determine the parameters of the reflected underwater shock wave as a function of the pressure on the front of the direct wave.

It can be shown that, as in the case of aerial shock waves, the ratio $(p_{otr} - p_0)/(p_f - p_0)$ is always larger than two, and only for weak waves does it tend to this value.

If it is necessary to calculate the reflection pressure for a wider range, it is necessary to assume in lieu of Eq. (1.201) a dynamic-compatibility condition in the form (1.198) and to carry out variations of similar nature.

Such calculations were carried out by several authors and yielded close results. Some of the calculation results are given in Table 4.

For some values of the pressures on the front of the direct wave we give here the ratios of the pressures in the reflected wave to the pressures in the direct wave, p_{otr}/p_f.

TABLE 4

1 По данным	2 Величины давления на фронте прямой волны, кг/см²					
	500	1000	5000	10 000	25 000	50 000
3 Пенни и Дас Гупта	2,088	2,170	2,60	2,92	3,44	3,93
4 Сунцова и Патрашева	2,040	2,15	2,55	2,88	3,35	3,74
5 По динамической адиабате (1.198)	2,055	2,11	2,36	2,60	3,05	3,50

1) Data by; 2) values of the pressure on the front of the direct wave, kg/cm²; 3) Penney and Das Gupta; 4) Suntsov and Patrashev; 5) according to the dynamic adiabatic curve (1.198).

In conclusion it must be emphasized that for an underwater explosion, the nonlinear problem of direct incidence of the shock wave on an absolutely rigid partition is essentially of theoretical interest, since the coefficient of reflection deviates appreciably from two at pressures for which even such materials as steel must be regarded as elastic, and consequently the assumption that the wall is absolutely rigid becomes an unrealistic one.

Such calculations are of practical significance only in the study of the interaction between shock waves of equal intensity.

Example 1. Determine the reflection pressure and the temperature on the front of the reflected aerial wave for a normally incident direct wave with excess pressure $\Delta p_f = 5.0$ kg/cm² on a rigid wall.

Solution. From Formula (1.187) we obtain the temperature on the front of the direct wave:

$$T_\phi = T_0 \frac{\frac{k+1}{k-1} + \frac{p_\phi}{p_0}}{\frac{k+1}{k-1} + \frac{p_0}{p_\phi}};$$

$$T_\phi = 288 \frac{6 + 5,82}{6 + 0,172} = 552 \,°K.$$

With the aid of (1.220) we calculate the reflection pressure:

$$\frac{p_{отр} - p_0}{p_ф - p_0} = 2 + \frac{\frac{k+1}{k-1}(p_ф - p_0)}{(p_ф - p_0) + \frac{2k}{k-1}p_0};$$

$$\frac{p_{отр} - 1,033}{5,0} = 2 + \frac{6 \cdot 5,0}{5,0 + 7 \cdot 1,033} = 2,46;$$

$$p_{отр} = 5,0 \cdot 4,46 + 1,033 = 23,0 \ \text{kg/cm}^2.$$

We determine the ratio ρ_{otr}/ρ_f. On the basis of (1.222) we have

$$\frac{\rho_{отр}}{\rho_ф} = \frac{k\frac{p_ф}{p_0}}{(k-1)\frac{p_ф}{p_0} + 1};$$

$$\frac{\rho_{отр}}{\rho_ф} = \frac{1,4 \cdot 5,82}{0,4 \cdot 5,82 + 1} = 2,45.$$

Finally we obtain

$$\frac{T_{отр}}{T_ф} = \frac{\rho_{отр}}{p_ф}\frac{p_ф}{\rho_{отр}};$$

$$T_{отр} = 552\frac{23,0}{6,033}\cdot\frac{1}{2,45} = 860 \ °K.$$

Example 2. Determine the reflection pressure on the front of an underwater shock wave for a direct wave with pressure on the front $p_f = 600 \ \text{kg/cm}^2$ normally incident on an absolutely rigid wall.

Solution. The problem reduces to solving the system (1.217), (1.225), (1.226)

$$\frac{p_{отр} - p_0}{p_ф - p_0} = \frac{\frac{\rho_{отр}}{\rho_0} - 1}{\frac{\rho_{отр}}{\rho_ф} - 1}; \tag{1.217}$$

$$\frac{\rho_{отр}}{\rho_0} = \left[\frac{p_{отр} + B}{p_0 + B}\right]^{\frac{1}{n}}; \tag{1.225}$$

$$\frac{\rho_{отр}}{\rho_ф} = \left[\frac{p_{отр} + B}{p_ф + B}\right]^{\frac{1}{n}}. \tag{1.226}$$

The corresponding calculations are summarized in Table 5 and are presented in Fig. 28. On the basis of the obvious graphic constructions we arrived at the conclusion that the reflection pressure in this case amounts to 1275 kg/cm^2.

The coefficient $(p_{otr} - p_0)/(p_f - p_0) = 2.130$ is merely 6.5% different from the acoustic coefficient.
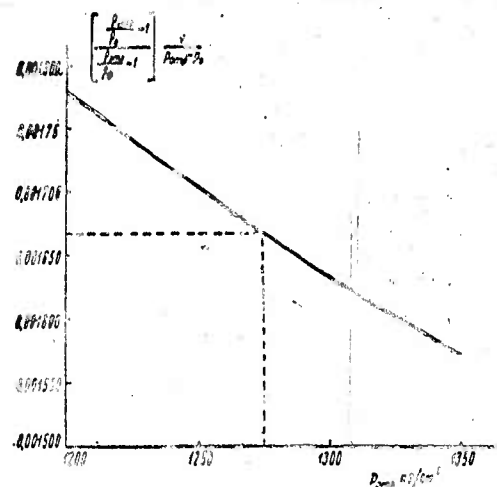
Fig. 28. Illustrating Example 1 of §11.

TABLE 5

$p_{отр}$	$\dfrac{p_{отр}-p_0}{p_ф-p_0}$	$p_{отр}+B$	$\dfrac{p_{отр}+B}{p_0+B}$	$\dfrac{p_{отр}+B}{p_ф+B}$	lg (4)	$\dfrac{1}{7,15}$ (6)
1	2	3	4	5	6	7
1 200	2,0001	4245	1,3936	1,1646	0,14426	0,020176
1 250	2,0851	4295	1,4100	1,1783	0,14922	0,020869
1 300	2,1669	4345	1,4264	1,1917	0,15412	0,021555
1 350	2,2520	4395	1,4428	1,2057	0,15927	0,022275

Continuation

lg (5)	$\dfrac{1}{7,15}$ (8)	(7) $\dfrac{1}{7,15}$	(9) $\dfrac{1}{7,15}$	$\dfrac{(10)-1}{(11)-1}$	(12) $\dfrac{1}{p_{отр}-p_0}$
8	9	10	11	12	13
0,06633	0,009277	1,047	1,022	2,1363	0,001779
0,07115	0,009951	1,049	1,023	2,1304	0,001704
0,07628	0,010668	1,051	1,024	2,1250	0,001634
0,08117	0,011364	1,053	1,025	2,1200	0,001570

§12. REFLECTION OF ACOUSTIC WAVES

In the preceding sections of the book we have emphasized several times that the laws governing the propagation of weak shock waves can be investigated by assuming that the entropy of the medium does not change.

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In this connection, great interest is attached to the limiting case of incidence of a wave of infinitesimally small amplitude on a wall. In such a formulation, the wave reflection problem is usually called the acoustic approximation problem.

We shall dwell later on in sufficient detail on the main premises of acoustics. For the time being we note that one such premise is the assumption that the propagation velocity of the perturbations is constant.

Assume that at some instant of time a plane acoustic wave is incident on an absolutely rigid surface at the point O (Fig. 29), the front of the wave making an angle α with the boundary.

The wave imparts to the gas a velocity whose normal component is v cos α. By virtue of the absolute rigidity of the boundary AOB, the normal component of the velocity on the boundary should be equal to zero.

This is attained only in the case when one superimposes on the existing velocity distribution a second wave front, which has at the point O such an intensity and such a direction, that its passage through the liquid following the front causes the formation of a normal velocity equal in magnitude and opposite in direction to the velocity produced by the direct wave along the line AO.

In other words, the following relation should be satisfied:

$$v \cos a - v' \cos a' = 0 \qquad (1.227)$$

In the case of small perturbations, both the direct and the reflected waves propagate with a velocity equal to the velocity of sound a_0. Then the velocity of displacement of the point O should obviously be equal to $a_0/(\sin \alpha)$ on one side and $a_0/(\sin \alpha')$ on the other. Consequently,

$$\frac{a_0}{\sin a} = \frac{a_0}{\sin a'}. \qquad (1.228)$$
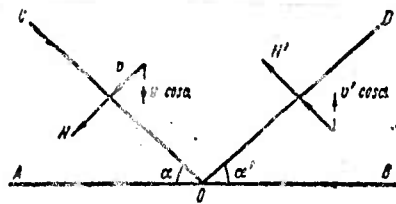
Fig. 29. Scheme of re-
flection of acoustic
waves.

This equation leads to the well-known law of acoustic reflection, according to which $\alpha = \alpha'$. But then also $v' = v$. Since the pressure on the front of the shock wave is determined by the formula $p = \rho_0 a_0 v$, by virtue of the equality of v and v' the pressure on the rigid wall is always equal to 2p, regardless of the direction of the direct shock wave.

Only when $\alpha = 90^\circ$ is there no reflection of the wave. The wave then glides along the wall and the pressure on the wall is equal to p.

The above-formulated consequences of the law of reflection of waves with infinitesimally small amplitude is known under the name acoustic paradox.*

Of great interest is the case when the acoustic wave is reflected not from an absolutely rigid wall, but from the boundary separating two compressible media. Here it is necessary to consider in addition to the direct and reflected waves also the so-called refracted wave, which propagates in the medium from the boundary of which the reflection takes place.

Just as in the preceding case, we can state that the velocity of displacement of the point 0 over the separation surface will charac-terize to an equal degree the motions of the direct, reflected, and refracted waves, i.e.,

$$\frac{a_0}{\sin \alpha} = \frac{a_0}{\sin \alpha'} = \frac{a''}{\sin \alpha''}.$$
(1.229)

As in the first case, the reflection law $\alpha = \alpha'$ holds here, too.

In addition, we get from (1.229) the so-called Snell's law for the determination of the direction of the refracted wave:

$$\frac{\sin \alpha}{\sin \alpha''} = \frac{a_0}{a''}.$$
(1.230)

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The continuity of motion and of the velocity of motion of the particles on the separation surface calls for satisfaction of the equation

$$v \cos a - v' \cos a' = v'' \cos a'', \atop p + p' = p''$$ (1.231)

Obviously, also,

$$p = \rho_0 a_0 v, \atop p' = \rho_0 a_0 v' \atop p'' = \rho_0'' a'' v''$$ (1.232)

The product $\rho_0 a_0$, which is the coefficient of proportionality between the pressure and particle velocity, is frequently called the acoustic impedance of the medium.

From Eqs. (1.231) and (1.232) it follows that

$$\frac{p_0}{\rho_0 a_0} - \frac{p'}{\rho_0 a_0} = \frac{p''}{\rho'' a''} \cdot \frac{\cos a''}{\cos a}$$

or

$$p - p' = p'' \frac{\rho_0 a_0}{\rho'' a''} \frac{\cos a''}{\cos a}.$$ (1.233)

Recognizing that $p'' = p + p'$, we obtain

$$\frac{p''}{p} = \frac{2\rho'' a'' \cos a}{\rho'' a'' \cos a + \rho_0 a_0 \cos a''};$$ (1.234)

$$\frac{p'}{p} = \frac{\rho'' a'' \cos a - \rho_0 a_0 \cos a''}{\rho'' a'' \cos a + \rho_0 a_0 \cos a''}.$$ (1.235)

Using Snell's law, Formulas (1.234) and (1.235) are readily reduced to the form

$$\frac{p''}{p} = \frac{2\rho'' a'' \cos a}{\rho'' a'' \cos a + \rho_0 a_0 \sqrt{1 - \left(\frac{a''}{a_0}\right)^2 \sin^2 a}};$$ (1.236)

$$\frac{p'}{p} = \frac{\frac{\rho'' a''}{\rho_0 a_0} \cos a - \sqrt{1 - \left(\frac{a''}{a_0}\right)^2 \sin^2 a}}{\frac{\rho'' a''}{\rho_0 a_0} \cos a + \sqrt{1 - \left(\frac{a''}{a_0}\right)^2 \sin^2 a}}.$$ (1.237)

The convenience of the last relations lies in the fact that they contain in addition to the acoustic impedances of the media only the angle of incidence of the direct wave.

If $a'' < a_0$, then reflected and refracted waves are produced for all angles of incidence of the direct wave.

The situation is different if $a'' > a_0$. In this case the reflection coefficients are real only in a definite range of angles of incidence

$$0 \leqslant \alpha \leqslant \alpha_1 = \text{arc} \sin \frac{a_0}{a''}. \qquad (1.238)$$

The angle $\alpha_1 = \text{arc} \sin a_0/a''$ is called the angle of total internal reflection.

For angles of incidence exceeding α_1, the reflection coefficient p'/p becomes complex with a modulus equal to unity. This means that the energy flux through the boundary separating two media is zero. No refracted wave is formed. The wave perturbations occurring at the separation boundary rapidly attenuate in amplitude with increasing depth of penetration into the second medium.*

It follows from Formula (1.235) that no reflected wave is formed only when the acoustic properties of the media are identically equal.

Indeed, equating the numerator of (1.235) to zero, we find that when the angle of incidence is

$$\alpha = \text{arc} \cos \frac{\rho_0}{a''} \sqrt{\frac{a_0^2 - a_0'^2}{\rho''^2 - \rho_0^2}}, \qquad (1.239)$$

we have

$$p'/p = 0.$$

Such a case is possible if $a_0 > a''$ and $\rho'' > \rho_0$, or $a_0 < a''$ and $\rho'' < \rho_0$.

In the case of direct incidence of a wave along a normal to the surface $\alpha = \alpha' = 0$ and Formulas (1.236)-(1.237) assume a particularly simple form

$$\frac{p''}{p} = \frac{2\rho''a''}{\rho''a'' + \rho_0 a_0}; \qquad (1.240)$$

$$\frac{p'}{p} = \frac{\rho''a'' - \rho_0 a_0}{\rho''a'' + \rho_0 a_0}. \qquad (1.241)$$

Relations (1.236)-(1.241) enable us to draw several important con-

clusions of physical nature:*

1. Upon reflection of a shock wave propagating in water from a medium with acoustic impedance that is lower than the acoustic imped- ance of water, a rarefaction wave is produced ($p'/p < 0$). The pressure on the front of the refracted wave is in this case lower than the pres- sure on the front of the direct wave. Consequently, the resultant pres- sure on the separation boundary is also lower than the pressure on the front of the direct wave.

2. Upon reflection of the same shock wave from a medium whose acoustic impedance is larger than the acoustic impedance of water, the reflected wave is a compression wave and the pressure on the front of the refracted wave will be larger than the pressure on the front of the direct wave.

Of great practical importance is the particular case of reflec- tion of an underwater shock wave from a free surface bordering on the atmosphere. Since the density of water is 775 times larger than the density of air, and the sound propagation velocity in water exceeds the sound propagation velocity in air by 4.4 times, the ratio of the acoustic impedances of water and air amounts to about 3400. Conse- quently, upon reflection of an underwater shock wave from a free sur- face, a rarefaction wave should arise, capable not only of completely suppressing the pressure in the direct wave, but also of causing ten- sile stresses in the liquid. The pressures on the front of the re- fracted aerial shock wave will then be negligibly small compared with the pressures on the front of a direct wave.

To the contrary, when an aerial shock wave strikes a free liquid surface, an underwater shock wave will arise, with an amplitude which is at least twice as large as the amplitude of the direct wave. Reflec- tion of the shock wave of an underwater explosion from the free sur-

face will be considered in detail subsequently.

Example. Calculate the resultant pressure on the wall of a wooden berth when an underwater shock wave with pressure on the front $p_f = 80$ atm is normally incident. The density of wood is assumed to be $\rho'' = 50$ kg-sec^2/m^2, and the velocity of sound in wood is $a'' = 1600$ m/sec.

Solution. The resultant pressure on the wall is equal to the pressure in the refracted wave.

We carry out the calculation in accordance with Formula (1.240):

$$\frac{p''}{p} = \frac{2\rho''a''}{\rho''a'' + \rho_0 a_0} = \frac{2 \cdot 50 \cdot 1600}{50 \cdot 1600 + 102 \cdot 1400} = 0,698 \ \text{kg/cm}^2.$$

We obtain

$$p'' = 0,698 \cdot 80 = 55,8 \ \text{atm}.$$

§13. LINEAR (REGULAR) REFLECTION OF SHOCK WAVES FROM A PLANE SEPARATION BOUNDARY BETWEEN TWO MEDIA

Let us consider now the reflection of a plane shock wave from an infinite wall. We first study a case when the wall is absolutely rigid. Let the angle of incidence of the direct wave be α (Fig. 30). The vanishing of the normal component of the particle velocity on the wall leads to the need for satisfying the relation

$$v_\phi \cos \alpha - v_{\text{отр}} \cos \alpha' = 0. \tag{1.242}$$

The velocity of displacement of the point O can be calculated with the aid of the dependence

$$\frac{N}{\sin \alpha} = \frac{N_{\text{отр}}}{\sin \alpha'}. \tag{1.243}$$

The only difference from the previously considered case is that now we have $N \neq N_{otr}$ and the shock-wave displacement velocities differ from the velocity of sound a_0 in the unperturbed medium.

The two equations (1.242) and (1.243) contain six variables: v_f, α, N_f, v_{otr}, α', N_{otr}. However, in this case, by virtue of the dynamic compatibility conditions, the parameters v_{otr} and N_{otr}, and also v_f

Fig. 30. Scheme of
regular reflection of
shock waves.

and N_f, are uniquely related with one another.

Thus, if we know the parameters of the direct wave, we have at our disposal two equations for the determination of the two variables v_{otr} (or N_{otr}) and α'.
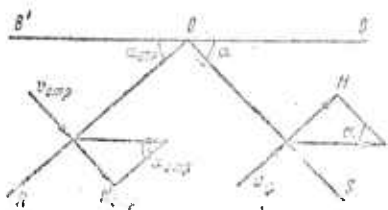
Numerical solution of these equations for a specified intensity of the direct wave and angle of incidence α yields two values of the angle α', and consequently two values of the corresponding quantities v_{otr} and N_{otr}.

These last parameters determine two values of p_{otr}.

Observations show that usually one realizes reflection with a smaller angle α', and in the general case $\alpha \neq \alpha'$. Only in the limiting case of infinitesimally weak waves are the angles of reflection and incidence equal. This case was considered in detail in the preceding section.

Fig. 31. Dependence of
the angle of reflection
on the angle of inci-
dence for different
pressures on the front
of an aerial shock wave.
1) Limiting.

Figure 31 shows plots of the calcu-
lated reflection angles for an ideal gas,
where the parameter is chosen to be the
ratio $\xi = p_0/p_f$.

From this figure it follows that the
larger the angle of incidence, the closer,
for a specified constant-pressure curve,
are the points that characterize two val-
ues of the angle of reflection α'.

At a certain angle $\alpha = \alpha_{pred}$ these
points coalesce. When $\alpha > \alpha_{pred}$, linear
reflection, a characteristic feature of which is the existence of one
common point for the direct and reflected waves lying on the boundary

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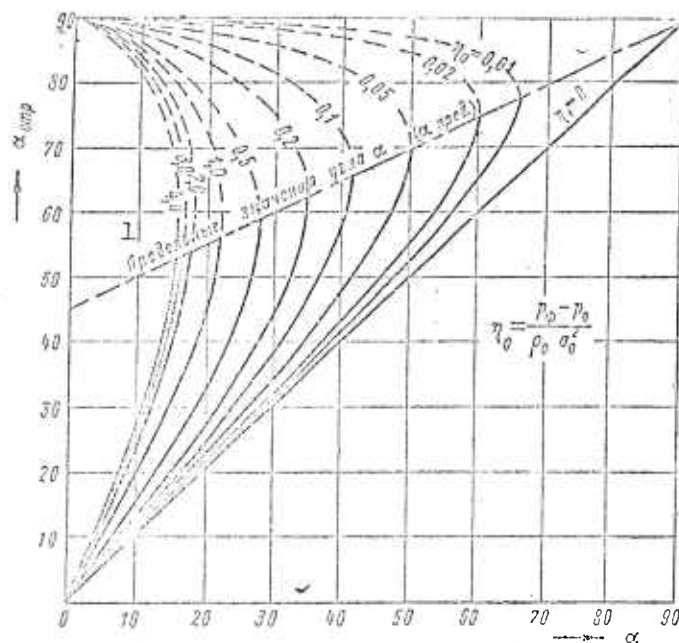
Fig. 32. Dependence of the angle
of reflection on the angle of in-
cidence for different pressures on
the front of an underwater shock
wave. 1) Limiting values of the
angle α.

surface, becomes impossible.

The phenomenon of linear reflection was investigated in great de-
tail by Neumann, Polyachek, Sieger, and A.N. Patrashev.

At small incidence angles, when the inequality $\alpha < \alpha^*$ obtains,
where for an ideal gas $\alpha^* = \frac{1}{2}\arccos\frac{k-1}{2} \simeq 40°$, the reflection angle is
smaller than the angle of incidence ($\alpha' < \alpha$) and the ratio p_{otr}/p_f is
smaller than for normal reflection, i.e., smaller than calculated by
the Izmaylov formula.

When $\alpha > \alpha^*$, the angle of reflection is greater than the angle of
incidence ($\alpha' > \alpha$) and p_{otr}/p_f is larger than for normal incidence.

However, linear reflection is possible for $\alpha > \alpha^*$ only if $\alpha^* <$
$< \alpha_{pred}$.

Calculations of similar nature were carried out also for water.
Their results are presented on Figs. 32 and 33. Figure 32, plotted
from the solution of A.N. Patrashev, gives the dependence of α' on α

for constant values of the pressure on the front.

Unlike air, in water the angle of reflection is always larger than the angle of incidence α, and consequently oblique reflection of a shock wave in water always yields a greater pressure than normal reflection, for identical pressures in the direct waves.

The coefficient ξ, which characterizes the pressure ratio in the reflected and direct waves, $\xi = (p_{otr} + B)/(p_f + B)$, is shown in Fig. 33 as a function of the angle of incidence for different pressures on the front of the direct wave.

It is easy to note that at pressure amplitudes in the direct wave below 600 atm the coefficient ξ differs little from unity, so that the pressure on a rigid wall practically doubles at incidence angles smaller than α_{pred}.

It is also of interest to investigate the more general case of reflection of shock waves from the separation boundary between two media, without assuming absolute rigidity of the wall.

Such a problem was considered for both normal and inclined incidence by A. I. Gubanov.*

In the case of normal incidence, using the notation of §12, the boundary conditions on the wall can be written in the form

$$v - v' = v''; \qquad (1.244)$$

$$p + p' = p''. \qquad (1.245)$$

The only difference from the acoustic approximation lies in the fact that the connection·between the pressures and velocities of the particles is expressed not by the linear relations (1.232), but in terms of the dynamic-compatibility conditions.

If, for example, we assume both the first and the second medium to be an ideal gas, then we obtain on the basis of (1.189)
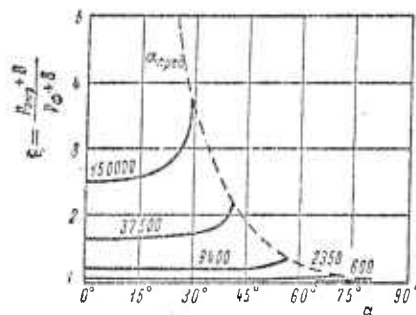
Fig. 33. Plot of
$\alpha_{pred} = f(\varepsilon, \lambda)$.
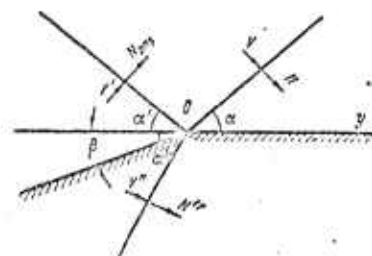
Fig. 34. Scheme of regular reflection of a shock wave from a surface separating two media.

$$v = \frac{a_0 \left(\frac{p}{p_0} - 1 \right)}{k \sqrt{\frac{k+1}{2k} \frac{p}{p_0} + \frac{k-1}{2k}}} ; \qquad (1.246)$$

$$v' = \frac{a \left(\frac{p'}{p} - 1 \right)}{k \sqrt{\frac{k+1}{2k} \frac{p'}{p} + \frac{k-1}{2k}}} ; \qquad (1.247)$$

$$v'' = \frac{a_0 \left(\frac{p''}{p_0} - 1 \right)}{k'' \sqrt{\frac{k''+1}{2k''} \frac{p''}{p_0} + \frac{k''-1}{2k''}}} . \qquad (1.248)$$

Consequently, assuming the parameters on the front of the direct wave known, we can always calculate the parameters of both the reflected and the refracted waves.

In the case of inclined incidence, the formulation of the boundary conditions become somewhat more complicated, for unlike the acoustic approximation, it is necessary here to take into account the angle that characterizes the deformation of the wall under the influence of the shock wave (Fig. 34).

The condition for a common point of the direct, reflected, and refracted waves on the surface separating the media will as before be

$$\frac{N}{\sin \alpha} = \frac{N_{отр}}{\sin \alpha'} = \frac{N''}{\sin \alpha''} . \qquad (1.249)$$

The equation for the pressures on the contact surface is written in the form

$$p + p' = p''. \tag{1.250}$$

The equation for the normal components of the particle velocities will be

$$v \cos(\alpha - \beta) - v' \cos(\alpha' + \beta) = v'' \cos(\alpha'' - \beta). \tag{1.251}$$

Finally, obviously, the point O is common not only for the wave fronts, but also for the branch of the contact surface OK_1.

The displacement velocity of the latter is equal to

$$v' \cos(\alpha + \beta).$$

Its velocity of motion along the y axis will be

$$\frac{v' \cos(\alpha + \beta)}{\sin \beta}.$$

The last ratio is equal to any of the ratios (1.249).

We thus have five equations for the determination of the nine unknowns: N_{otr}, α', v', p', N'', α'', v'', p'', β (the values of N, α, p, and v on the front of the direct wave are assumed specified).

In addition, from the dynamic compatibility conditions we can regard the following as known:

$$N_{\text{отр}} = N(p'),$$
$$v' = v(p'),$$
$$N'' = N(p''),$$
$$v'' = v(p'').$$

Thus, we again arrive at a closed system of equations, which when solved enables us to calculate all the parameters of the reflected and refracted waves.

Such calculations for the particular cases of strong and weak shock waves, and also for media that differ from each other greatly and little have been carried through by A. I. Gubanov to the stage of final approximate relations.

As was already noted above, when the angle of incidence of the direct waves are larger than the limiting angle, linear reflection be-

comes impossible. There occurs the so-called nonlinear or irregular reflection, which will be analyzed in the next section.

§14. NONLINEAR (IRREGULAR) REFLECTION OF SHOCK WAVES FROM A RIGID WALL

At angles of incidence larger than α_{pred}, as was already mentioned earlier, linear reflection becomes impossible. The reflected wave, propagating through the perturbed medium, overtakes the direct wave and, merging with the latter, forms a third wave frequently called the frontal shock wave or the Mach wave. A three-wave configuration results, a characteristic feature of which is the existence of a common point of intersection of the waves (the so-called triple point).
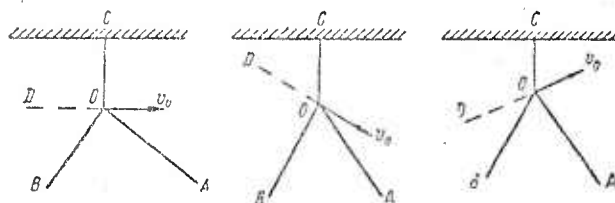
Fig. 35. Stationary, direct, and inverted nonlinear reflections.

Starting from the conditions of dynamic compatibility, we can show that the intersection of three strong discontinuity fronts is possible only if an additional condition is satisfied, consisting in the fact that another surface of stationary strong discontinuity must pass through the point O (Fig. 35). It is usually called the contact surface.

Depending on the location of the contact surface, a distinction is made between three types of nonlinear reflections (Fig. 35): stationary, direct, and inverted.

In stationary nonlinear reflection, the contact surface OD is parallel to the wall. In the case of direct nonlinear reflection, the contact surface is directed away from the wall; in the case of inverted reflection it is directed toward the wall.
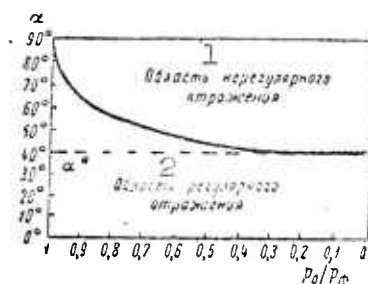
Fig. 36. Regions of
regular and irregu-
lar reflection. 1)
Region of irregular
reflection; 2) re-
gion of regular re-
flection.

Inverted nonlinear reflection is un-
stable, collapses, and can be disregarded.

Direct nonlinear reflection is realized
most frequently and is of greatest practical
interest.

Figure 36 shows the curve for an ideal
gas, representing the dependence of α_{pred} on
the ratio of the pressure in the unperturbed
medium p_0 to the pressure in the direct wave
p_f.

This curve divides the region of all possible values of the angle
α and the ratio p_0/p_f into two parts: a region where linear reflection
is possible, and a region of existence of nonlinear reflection. Start-
ing with $p_0/p_f \simeq 0.5$ and up to $p_0/p_f = 0$ the limiting angle amounts to
about $40°$.

Problems in the theory of irregular reflection, following Mach,
engaged many researchers. However, the complexity of the boundary con-
ditions of the problem have made it necessary to introduce definite
simplifying assumptions, and consequently at the present time the re-
sults of the theory are still far from complete and in many cases
yield serious discrepancies from the experimental data. A practical
estimate of the parameters of the frontal wave is therefore carried
out with the aid of empirical relationships.

A more detailed exposition of irregular reflection of an aerial
shock wave from the surface of the earth (or water) will be given later
on. As to irregular reflection of shock waves in the case of an under-
water explosion, by virtue of the considerations presented in §11,
such a reflection is rarely realized and therefore this problem is not
considered in the book.

[Footnotes]

9   By homogeneous system we shall henceforth mean an aggregate of elementary particles which are identical in their chemical and physical properties.

16   b,m,v,p, — infinitesimals of higher order.

19   Here and elsewhere in this section we denote by ε_i a quantity of first order of smallness.

19   The minus sign in the term containing c_n is the consequence of the fact that the direction of the outward normal to the surface S_1 does not coincide with the direction of motion of the discontinuity surface.

20   In the literature such a surface is frequently called a contact surface, or a surface of tangential discontinuities.

21   If $[p] = 0$, then from (1.40) $[v] = 0$, and consequently, $[\theta] = 0$ and $[\rho] = 0$, i.e., there is no strong discontinuity whatever.

21   The variables θ and u are not independent, since by definition $\theta = N - v_n$, and the value of the internal energy is expressed in terms of p and ρ with the aid of the equation of state.

25   For further details see Ya.B. Zel'dovich "Shock-Wave Theory and Introduction to Gasdynamics," Acad. Sci. USSR Press, 1946.

33   Formula (1.69) follows from the general expression

$$\operatorname{grad}(\bar{a}, \bar{b}) = (\bar{b}_\nabla)\bar{a} + (\bar{a}_\nabla)\bar{b} + \bar{b}\times\operatorname{rot}\bar{a} + \bar{a}\times\operatorname{rot}\bar{b},$$

if we put $\bar{a} = \bar{b} = \bar{v}$ (see, for example. N.Ye. Kochin, Vector Analysis and Introduction to Tensor Analysis, GONTI, 1938).

42   Having defined the characteristics in this manner, we shall leave out henceforth the index L.

45   Unlike the previous coordinates r_i and t_i, here r'_i and t'_i characterize the streamline (the trajectory of motion of a liquid particle).

53   See, for example, Koshlyakov, Fundamental Differential Equations of Mathematical Physics, GTTI, 1933.

54   Relationship (1.163) can be derived, by considering directly the equations of one-dimensional isentropic motion of a liquid with plane symmetry.

    Indeed, putting $\ln \theta = \text{const}$ and $\nu = 1$ in Formulas (1.97) and (1.98), we obtain

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial r} + \frac{2a}{k-1} \frac{\partial a}{\partial r} = 0; \qquad (1.164)$$

$$\frac{2}{k-1} \left(\frac{\partial a}{\partial t} + v \frac{\partial a}{\partial r} \right) + a \frac{\partial v}{\partial r} = 0. \qquad (1.165)$$

    Subtracting and adding these equations, we can write them in the form

$$\frac{\partial}{\partial t} \left(v + \frac{2}{k-1} a \right) + (v + a) \frac{\partial}{\partial r} \left(v + \frac{2}{k-1} a \right) = 0,$$

$$\frac{\partial}{\partial t} \left(v - \frac{2}{k-1} a \right) + (v - a) \frac{\partial}{\partial r} \left(v - \frac{2}{k-1} a \right) = 0.$$

    Obviously, a possible solution of the first equation will be $v + 2a/(k - 1) = \text{const}$, that of the second is $v - 2a/(k - 1) = \text{const}$, in agreement with the result (1.163).

62
$$\cdot\, a_0^2 = \frac{dp}{d\rho} \Big|_{\rho=\rho_*} \qquad dp = n (p_* + B) \left(\frac{\rho}{\rho_*} \right)^n \frac{1}{\rho_*} d\rho,$$

whence

$$a_0^2 \frac{dp}{d\rho} \Big|_{\rho=\rho_*} = n (p_* + B) \frac{1}{\rho_*} .$$

67   See, for example, L.D. Landau and Ye.M. Lifshits, Mechanics of Continuous Media, GITL, 1953.

71   At low pressures on the wave front, Formulas (1.181)-(1.189) can be linearized. In particular, if $\Delta p_f / p_0 < 1$ we have with error not larger than 5%

$$N = a_0 \left(1 + \frac{k+1}{4k} \frac{\Delta p_\Phi}{p_0} \right); \qquad (1.190)$$

$$a_\Phi = a_0 \left(1 + \frac{k-1}{2k} \frac{\Delta p_\Phi}{p_0} \right). \qquad (1.191)$$

71   As the initial data for the calculation we have assumed the parameters of the international atmosphere: $a_0 = 340$ m/sec, $p_0 = 1.033$ kg/cm^2, $\rho_0 = 0.125$ kg-sec^2/m^4, $T_0 = 15^\circ$C, and $k = 1.40$.

75   See, for example, R. Koul, Underwater Explosions, IIL, 1950.

[Footnotes (Continued)]

76   Obviously, knowing the values of the pressures and densities on the front of the wave, it is also easy to calculate the temperature from the equations of state. However, such a formula is cumbersome, and is therefore not presented here. For the same reason we have also left out the expression for the velocity of sound, which can be obtained from the adiabaticity conditions of water.

80   Equations (1.208)-(1.211) were derived by N.N. Suntsov. The coefficient m was assumed by him to be equal to 2.1.

85   If the direct wave is stationary, then the reflected wave will also be stationary, and the particle velocities will be equal to zero not only on the wall but everywhere behind the front of the reflected wave.

93   Actually, for waves of finite amplitude, at a certain angle of incidence $\alpha < 90°$, as will be discussed in detail later on, this type of reflection becomes impossible and is replaced by the so-called nonlinear (irregular) reflection.

95   A detailed analysis of the case of total internal reflection of acoustic waves with an estimate of the depth of penetration of the wave perturbations as a function of the length and frequency of the direct wave is given in the papers of L.M. Brekhovskikh. See, for example, L.M. Brekhovskikh, Waves in Layered Media, Acad. Sci. USSR Press, 1957.

96   Strictly speaking, the first derivation is valid only for angles of incidence smaller than the angle of total internal reflection. Usually, however, if $\rho''a'' < \rho_0 a_0$, then also $a'' < a_0$.

100   A.I. Gubanov, Reflection and Refraction of Shock Waves on a Boundary Between Two Media. ZhTF, Vol. 28, 1958 and 29, 1959.

[List of Transliterated Symbols]

10   внутр = vnutr = vnutrennyy = internal

10   внешн = vneshn = vneshnyy = external

28   ф = f = front = front

35   кр = kr = kriticheskiy = critical

63    $кг/км^2$ = kg/cm^2

63    $г/км^3$ = g/cm^3

63	м/сек = m/sec
71	ск = sk = skorostnoy napor = velocity head
73	кг/км^2 = kg/cm^2
73	м/сек = m/sec
73	кал/г-дег = cal/g-deg
85	отр = otr = otrazhennaya volna = reflected wave
86	атм = atm = atmosfernaya volna = atomospheric wave
98	пред = pred = predel'noye otrazheniye = limit reflection

Chapter 2

EXPLOSION IN AN UNBOUNDED MEDIUM

§1. FORMULATION OF EXPLOSION PROBLEM

As indicated above, in the general case the distribution of pressures and temperatures on the boundary separating the explosion products and the surrounding medium at the initial instant of time is arbitrary, since the character of a reaction such as an explosion does not depend on the properties of the surrounding medium. Using the previously adopted terminology, this surface would have to be regarded as a nonstationary strong-discontinuity surface. However, the dynamic compatibility conditions, which should be satisfied on a nonstationary strong-discontinuity surface, are not satisfied in this case, owing to the arbitrariness of the parameters of the explosion products. Consequently, it is impossible likewise for a single discontinuity surface to exist.

N.Ye. Kochin has proved that for all possible cases the initial surface breaks up into three surfaces. One of these surfaces, propagating in a medium surrounding the charge, is the nonstationary strong discontinuity surface (shock-wave front); the second, separating the explosion products from the medium, is a stationary strong-discontinuity surface (the gas-bubble surface); finally, the third, propagating with the explosion products, is a weak-discontinuity or characteristic surface.

The main problem of the theory of explosion in an unbounded liquid is the study of unsteady motion of a liquid between boundary sur-

faces, namely the front of the shock wave and the gas-bubble surface. Such unsteady motion is determined by the following system of equations:

$$\rho \frac{dv}{dt} + \text{grad}\, p = 0; \qquad (2.1)$$

$$\frac{\partial \rho}{\partial t} + \text{div}\, (\rho \vec{v}) = 0. \qquad (2.2)$$

This system contains three variables: pressure, density, and velocity, so that it is not a closed system.

In order to close this system, it is assumed in many investigations that the solution of the problem can be obtained by assuming the liquid to be incompressible: $\rho = $ const. If the motion is in this case potential over the entire region of flow, then the matter reduces to an analysis of the well-studied Laplace equation $\Delta \varphi = 0$. This is indeed the procedure used in his time by Lamb, to whom belongs the first theoretical solution of the problem of underwater explosions.

However, the assumption that the medium in which the shock wave propagates is incompressible does not in fact correspond to reality. The point is that explosions are accompanied by considerable pressures, in which the water is appreciably compressed (for example, more than 20% at pressures of 10 thousand atm).

In addition, and this is the major point, the use of the hypothesis that liquid is incompressible does not make it possible to take into account the energy dissipation, which is rather great in the case of explosions.

For example, in an underwater explosion approximately 60% of the initial energy is dissipated during the first pulsation, while in an aerial explosion the fraction is even larger, something which obviously cannot be disregarded.

Finally, the incompressibility hypothesis excludes from considera-

tion shock waves and the local explosion phenomena as well as the finite perturbation propagation velocities which are connected with their existence.

In view of this, the incompressible liquid hypothesis cannot be used to study the explosion process. This indeed raises the need for setting up additional supplementary equations to close the system. Such equations are the equation of state and the energy equation, which contain, in addition to the foregoing, two other variables, namely the absolute temperature T and the heat influx ε. But in most cases the high speeds of the explosion processes practically exclude the possibility of heat exchange with the surrounding medium, so that the heat flux is assumed equal to zero ($\varepsilon = 0$) and the system of equations is closed.

However, solution of such a system of equations for the general case entails great mathematical difficulties. In view of this, only a series of investigations for motion with plane, axial, and spherical symmetry have been carried out. At the same time it must be noted that it is precisely the analysis of one-dimensional motion which is of primary interest.

Exact solution of the explosion problems has been obtained only for motions with plane symmetry. In all other remaining cases only approximate methods have been developed. We present here only some of these methods, paying principal attention not to the derivations and transformations, but to the research methods.

In calculations based on any of the existing schemes, it is necessary to know the hydrodynamic elements at the initial instant of time.

We now proceed to such an estimate.

Assume that at some instant of time, which we shall regard as in-

itial (t = 0), the values of the hydrodynamic elements v_x, ρ, and $\underline{p}$ are specified on both sides of the plane x = 0. We assume that for $r < 0$ we have at that instant everywhere

$$v_x = \text{const} = v_1; \quad p = \text{const} = p_1; \quad \rho = \text{const} = \rho_1$$

and for $r > 0$

$$v_x = \text{const} = v_2; \quad p = \text{const} = p_2; \quad \rho = \text{const} = \rho_2.$$
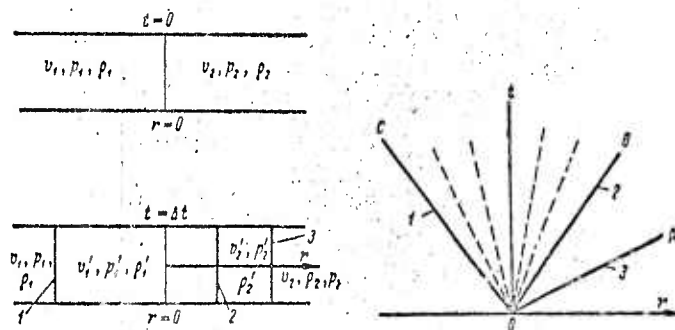
Fig. 37. Breakup of initial explosion surface. 1) Weak-discontinuity surface; 2) stationary strong-discontinuity surface; 3) nonstationary strong-discontinuity surface.

These six quantities are in general perfectly arbitrary. On the plane r = 0 the dynamic compatibility conditions are not satisfied, so that the initial discontinuity surface breaks up into three surfaces (Fig. 37). In the medium with parameters ρ_2, v_2, and p_2 there will propagate a nonstationary strong-discontinuity surface (shock wave) with some velocity N to be determined (line OA). Moving in the same direction will be a stationary strong-discontinuity surface, which separates one medium from the other (line OB). The velocity of its displacement, $\underline{v}$, is also to be determined. Finally, to the left of the initial surface there will propagate a weak-discontinuity surface (or characteristic) with velocity equal to the velocity of sound in the medium p_1, ρ_1 (line OC).

We shall assume the quantities v_1, v_2, p_1, p_2, ρ_1, ρ_2 known, with $v_2 = 0$, $p_2 = p_0$, and $\rho_2 = \rho_0$.

Then, to estimate the hydrodynamic elements v'_2, p'_2, ρ'_2, v'_1, p'_1, ρ'_1, N we shall have the following system of equations:

1) on the nonstationary strong-discontinuity surface

$$[\rho^0] = 0, \qquad (2.3)$$
$$\rho^0 [v] = [p]; \qquad (2.4)$$
$$\rho^0 \left[\frac{v^2}{2} + u \right] = [pv]; \qquad (2.5)$$

2) on the stationary strong-discontinuity surface

$$[v] = 0 \text{ or } v'_1 = v'_2 = v'; \qquad (2.6)$$
$$[p] = 0 \text{ or } p'_1 = p'_2 = p'. \qquad (2.7)$$

Since the entropy does not change on going through the weak-discontinuity surface, we have

$$\frac{p'_1}{\rho'^k_1} = \frac{p_1}{\rho^k_1} \qquad (2.8)$$

We note further that the line OC is a characteristic of the second family (rarefaction wave). On this characteristic the values of $\underline{v}$ and $\underline{a}$ are constant, in accordance with the conditions of the problem, and consequently the following relation will be satisfied in the entire region

$$v' + \frac{2}{x-1} a' = \frac{2}{x-1} a_1 + v_1. \qquad (2.9)$$

The system of equations (2.3)-(2.9) enables us to determine uniquely the seven unknown quantities.

For an ideal gas Eq. (2.5) is conveniently replaced by the Hugoniot adiabatic curve

$$\frac{p_2}{p_1} = \frac{(k+1)\rho_2 - (k-1)\rho'_2}{(k+1)\rho'_2 - (k-1)\rho_2}, \qquad (2.10)$$

and for water it is replaced by the dynamic adiabatic curve [see (1.198)]:

$$p' - p_0 = d \left[(\rho'_2)^{k_1} - \rho_0^{k_1} \right]. \qquad (2.11)$$

Let us now carry out the necessary analysis for estimating the hydrodynamic elements on the front of an underwater shock wave at the

initial instant of time. Neglecting p_0 we obtain from (2.11)

$$p' = d \left[\rho_2'^{\varkappa} - \rho_0^{\varkappa} \right],$$

(2.12)

but

$$\frac{p_1}{\rho_1^{\varkappa}} = \frac{p_1'}{\rho_1'^{\varkappa}},$$

hence

$$p_1' = \frac{p_1}{\rho_1^{\varkappa}} \rho_1'^{\varkappa}.$$

(2.13)

On the basis of (2.12) and (2.13) we have

$$\rho_1' = \rho_1 \left[d \left(\rho_2'^{\varkappa} - \rho_0^{\varkappa} \right) \right]^{\frac{1}{\varkappa}} p_1^{-\frac{1}{\varkappa}}.$$

(2.14)

Finally, from (2.9)

$$v' + \frac{2}{\varkappa - 1} a' = v_1 + \frac{2}{\varkappa - 1} a_1,$$

or

$$v' + \frac{2}{\varkappa - 1} \sqrt{\varkappa \frac{p'}{\rho_1'}} = v_1 + \frac{2}{\varkappa - 1} \sqrt{\varkappa \frac{p_1}{\rho_1}}.$$

(2.15)

Substituting the values of ρ'_1 and p' in (2.15) and recognizing that*

$$v' = \sqrt{d \left(\rho_2'^{\varkappa} - \rho_0^{\varkappa} \right) \left(\frac{1}{\rho_0} - \frac{1}{\rho_2} \right)},$$

(2.16)

we obtain finally

$$\sqrt{d \left(\rho_2'^{\varkappa} - \rho_0^{\varkappa} \right) \left(\frac{1}{\rho_0} - \frac{1}{\rho_2} \right)} + \frac{2}{\varkappa - 1} \sqrt{\frac{\varkappa}{\rho_1} \left[d \left(\rho_2'^{\varkappa} - \rho_0^{\varkappa} \right) \right]^{1 - \frac{1}{\varkappa}} p_1^{\frac{1}{\varkappa}}} =$$

$$= \frac{2}{\varkappa - 1} \sqrt{\varkappa \frac{p_1}{\rho_1}} + v_1.$$

(2.17)

Equation (2.17) contains only one unknown quantity, ρ'_2. Solving this equation we can calculate the density on the front of the shock wave at the initial instant of time. Then, using the dynamic compatibility conditions, we can readily obtain all the remaining hydrodynamic elements.

Perfectly analogous arguments enable us to obtain a corresponding dependence for the case of an ideal gas. Such a dependence has the form**

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$$\sqrt{p_0 \frac{(k+1)\frac{\rho_2'}{\rho_0} - (k-1)}{(k+1) - (k-1)\frac{\rho_2'}{\rho_0}} \left(\frac{1}{\rho_0} - \frac{1}{\rho_2'}\right)} +$$

$$+ \frac{2}{\varkappa - 1} \sqrt{\frac{\varkappa}{\rho_1} \left[p_0 \frac{(k+1)\frac{\rho_2'}{\rho_0} - (k-1)}{(k+1) - (k-1)\frac{\rho_2'}{\rho_0}} \right]^{1-\frac{1}{\varkappa}} p_1^{\frac{1}{\varkappa}}} =$$

$$= \frac{2}{\varkappa - 1} \sqrt{\varkappa \frac{p_1}{\rho_1}} + v_1. \qquad (2.18)$$

The unknown quantity is again the density on the front of the shock wave, ρ'_2.

In carrying out calculations by Formula (2.18), it must be borne in mind that usually the ratio ρ'_2/ρ_0 in problems of this type is close to the value of $(k + 1)/(k - 1)$. We can therefore put

$$\frac{\rho_2'}{\rho_0} \simeq \frac{k+1}{k-1} - (\alpha + \alpha^2),$$

where α is a small quantity of first order.

Here

$$\frac{\frac{k+1}{k-1}\frac{\rho_2'}{\rho_0} - 1}{\frac{k+1}{k-1} - \frac{\rho_2'}{\rho_0}} = \frac{\frac{4k}{(k-1)^2} - \frac{k+1}{k-1}\alpha(\alpha+1)}{\alpha(\alpha+1)}.$$

Calculation of the foregoing expression no longer entails an estimate of a difference between nearly equal quantities, and is consequently more convenient for practical purposes.

In Table 6 are given the values of the hydrodynamic elements on the front of a shock wave at the initial instant of an underwater and aerial explosion for four types of explosives. The data characterizing the detonation of the explosive are those of L. D. Landau and K. P. Stanyukovich.*

The results of calculations for an underwater explosion give an order of magnitude close to those obtained by experiment. Thus, in a study of explosions produced in water by a charge of PETN, Dering

TABLE 6

Hydrodynamic Elements at the Initial Instant of Explosion

1 Формула или обозначение	2 Взрывчатое вещество			
	3 тол	4 пикриновая кислота	5 тэн	6 тетрил
7 Параметры детонационной волны				
8 Давление на фронте детонационной волны, кг/см² (p₁)	190 000	210 000	275 000	230 000
9 Показатель адиабаты продуктов взрыва (х)	3,2	3,1	3,2	3,1
10 Плотность ВВ, кг сек²/м⁴ (ρ₁)	163	173	173	163
11 Скорость частиц продуктов взрыва, м/сек (v₁)	1 680	1 730	2 000	1 850
12 Скорость детонации, м/сек	7000	7100	8400	7600
13 Параметры подводной ударной волны				
14 Плотность воды на фронте, кг сек²/м⁴ (ρ'₂)	177	179	178	182
15 Давление на фронте, кг/см²	133 500	143 500	140 000	160 000
16 Скорость перемещения ударной волны, м/сек (N)	5 550	5 730	5 700	6 000
17 Скорость частиц за фронтом, м/сек (v'₂)	2370	2475	2450	2650
18 Температура на фронте, °C	590	640	610	725
19 Параметры воздушной ударной волны				
20 Плотность воздуха на фронте, кг сек²/м⁴ (ρ'₂)	0,7428	0,7437	0,7461	0,7445
15 Давление на фронте, кг/см²	622	756	970	838
16 Скорость перемещения ударной волны, м/сек (N)	7720	8530	9650	8960
17 Скорость частиц за фронтом, м/сек (v'₂)	5940	6590	7440	6920
21 Температура на фронте, °K	28 900	32 800	44 200	38 100

1) Formula or symbol; 2) explosive; 3) TNT;
4) picric acid; 5) PETN; 6) tetryl; 7)
parameters of detonation wave; 8) pressure
on the front of the detonation wave,
kg/cm^2 (p_1); 9) adiabatic exponent of explosion products (κ); 10) density of explosive, $kg\text{-}sec^2/m^4$ (ρ_1); 11) velocity of
the explosion-product particles, m/sec
(v_1); 12) velocity of detonation, m/sec;
13) parameters of underwater shock wave;
14) density of water on the front, $kg\text{-}sec^2$
per m^4 (ρ'_2); 15) pressure on the front,
kg/cm^2; 16) displacement velocity of the

TABLE 6 (Continued)

shock wave, m/sec (N); 17) velocity of par-
ticles behind the front, m/sec (v'_2); 18)
temperature on the front, $^{\circ}C$; 19) param-
eters of aerial shock wave; 20) density of
air on the front, kg-sec^2/m^4 (ρ'_2); 21) $^{\circ}K$.

obtained for the displacement velocity of the shock wave about 6800
m/sec and for the pressure 140,000 atm. B.I. Shekhter obtained for TNT
N = 6100 m/sec and p = 136,000 atm. The density jump registered upon
the explosion of a charge of PETN in water was $\rho_f/\rho_0 = 1.75$.*

It can be assumed that a somewhat greater discrepancy between
theory and experiment will be obtained for an aerial explosion, in
view of the considerable deviation of the equation of state of real
gases at high pressures and temperatures from the Mendeleyev-Clapeyron
equation of state.

If we assume k = 1.25, then we obtain for a TNT explosion:
air density on the front $\rho_f = 1.117$ kg-sec^2/m^4;
pressure on the wave front $p_f = 1370$ kg/cm^2;
displacement velocity N = 11,900 m/sec.

§2. LAWS OF SIMILARITY IN THE THEORY OF EXPLOSIONS

The study of the complicated processes that accompany an explo-
sion, both with theoretical methods and particularly with experimental
methods, is impossible without extensive use of the general laws of
similarity of physical phenomena. The theory of similarity and dimen-
sionality makes it possible to point to the most rational analysis
scheme and permits us to choose in most convenient form the minimum
number of parameters reflecting the many aspects of the investigated
process, and helps generalize the obtained experimental data.

The theory of physical similarity is based on three principal
theorems.

First theorem. If a group of phenomena defined by corresponding systems of equations and uniqueness conditions, is similar, then the quantities contained in the indicated systems should form complexes which retain one and the same numerical value for a specified group of phenomena. These complexes are called the similarity invariants.

In other words, the first theorem of similarity theory indicates that a physical phenomenon does not depend on the measurement scale and can be characterized fully by some dimensionless combinations of defining parameters.

The second theorem of similarity theory, sometimes called the π theorem, states that in any phenomenon it is possible to find connections not only between the real named quantities, but also their dimensionless combinations.

In this case, if there exist n dimensional quantities, characterizing the phenomenon, and the problem is considered in k fundamental units, then the number of defining similarity criteria is equal to $n - k$, where one criterion arbitrarily chosen among them can be regarded as a function of the remaining criteria.

Third theorem. The set of phenomena defined by specified differential equations and uniqueness conditions constitutes a similar system, if the quantities contained in the uniqueness conditions also constitute a similar system, and the invariants of this system, defined by the specified equations and setup of the indicated quantities, have one and the same numerical value.

In other words, the third theorem of similarity theory stipulates in its most general formulation the fulfillment of similarity in the boundary conditions of the problem.

In particular, it follows from the third theorem that systems that are similar prior to the start of some physical process, will re-

main similar also after the termination of this process, if the process itself corresponds to the similarity conditions.

In mechanics one makes a distinction between geometrically, kinematically, and dynamically similar systems.

Two systems are called geometrically similar if the ratio of their comparable linear dimensions is the same. The ratio of the linear dimensions l_1 of one system to the comparable dimensions l_2 of the other system is called the modulus or scale of geometrical similarity $\lambda = l_1/l_2$.

Geometrically similar systems can execute geometrically similar displacements. If the ratio of the time intervals during which the corresponding points of the systems describe comparable sections of trajectories is the same for all points of the system and is maintained constant during the entire time of motion, such systems are called kinematically similar.

Thus, kinematic similarity is defined by means of two scales: the linear scale λ and the time scale $\tau = t_1/t_2$.

Two systems are called dynamically similar if in addition to having kinematic similarity they are also similar in their mass distributions.

Thus, in the case of dynamic similarity there is added another scale, the ratio of the masses at the comparable points:

$$M = m_1/m_2.$$

On the basis of all the foregoing we can present the following definition of similarity.

Similar systems are ones in which all the quantities characterizing the state and motion of one system can be obtained by simple multiplication of the corresponding quantities of a different system by constant factors.

From an analysis of the Navier-Stokes equations it follows that
the motions of a compressible viscous liquid, subject to the action of
mass forces, are dynamically similar if the following four dimension-
less criteria are identically equal to one another at comparable points
of the flow:

the homochrony or Struchal number

$$H = vt/l; \qquad (2.19)$$

the Froude number

$$F = v^2/gl; \qquad (2.20)$$

the Euler number

$$E = p/\rho v^2; \qquad (2.21)$$

the Reynolds number

$$Re = vl/\nu; \qquad (2.22)$$

where v is the particle velocity, t the time, g the acceleration due
to gravity, l the characteristic linear dimension of the body, p the
pressure, ρ the density, and ν the kinematic viscosity coefficient.

In most cases the similarity conditions can be satisfied only in
part. It becomes necessary then to estimate the errors arising when
the results of model investigations are recalculated for the natural
phenomena (the so-called scale effect).

Inasmuch as in gasdynamics one studies the motion of an ideal
fluid which is not subject to the action of external forces, only two
out of the four similarity criteria of hydrodynamics need to be taken
into account, the Struchal number and the Euler number.

Lack of similarity after Froude and Reynolds in such problems can
obviously not give rise to a scale effect.

Let us assume that the natural and model explosions are carried
out in one and the same medium; we then have $\rho_{On} = \rho_{Om}$.

From the conditions of dynamic compatibility it follows that in-

dependently of the causes leading to the formation of the shock wave, there exists between the hydrodynamic elements on the front a one-to-one relationship. Consequently, if it is ascertained that at definite distances from the explosion centers of two charges the pressures on the front are identical, then the Euler numbers are likewise identically equal

$$\frac{p_{\text{н}}}{\rho_{\text{н}} v_{\text{н}}^2} = \frac{p_{\text{м}}}{\rho_{\text{м}} v_{\text{м}}^2}.$$

The simplest case from the point of view of similarity is when the hydrodynamic fields are investigated under explosions of charges made of the same explosive. Since the detonation waves arrive in this case at the charge surfaces with identical parameters, then complete similarity of the fields will obtain even in the immediate vicinity of the explosion center.

From the Struchal conditions, assuming that the particle velocities are equal to each other at the comparable points, it follows that

$$\frac{t_{\text{н}}}{l_{\text{н}}} = \frac{t_{\text{м}}}{l_{\text{м}}}.$$

This means that in the simulation of liquid motion due to an explosion, the scale of the linear dimensions should be the same as the time scale.

The characteristic linear dimensions that determine the scale of the phenomena in explosions are the dimensions of the charge. This leads to the following law of similarity in explosions of identical explosives: the parameters of the medium do not change in the motion caused by the explosion if the scales of length and time with which these parameters are measured are increased or decreased by the same factor as the dimensions of the charge.

For example, the pressure in the shock wave, measured at a distance r from a TNT charge of spherical form with radius R_{03} should be

equal to the pressure in the shock wave at a distance λr from a charge of radius λR_{03}.

The pressure patterns at the corresponding distances will have the form shown in Fig. 38, which illustrates clearly the deformation of the pattern pressure at comparable distances upon change in the charge dimensions.

The formulation of the similarity laws becomes somewhat more complicated upon comparison of the hydrodynamic fields produced in a liquid in explosions of charges of different explosives. Experience has shown that the similarity can be based in this case on the energy principle. It is natural that in the direct vicinity of the charge, there will be no similarity of the fields in such cases, in view of the differences in the boundary conditions. However, since the process of energy dissipation occurs at high speed in the case of strong shock waves, there unavoidably occurs an instant when at definite distances from the centers of two different charges the values of the pressures turn out to be identical, and consequently, by virtue of the conditions of dynamic compatibility, the pressures and the particle velocities are also equal.
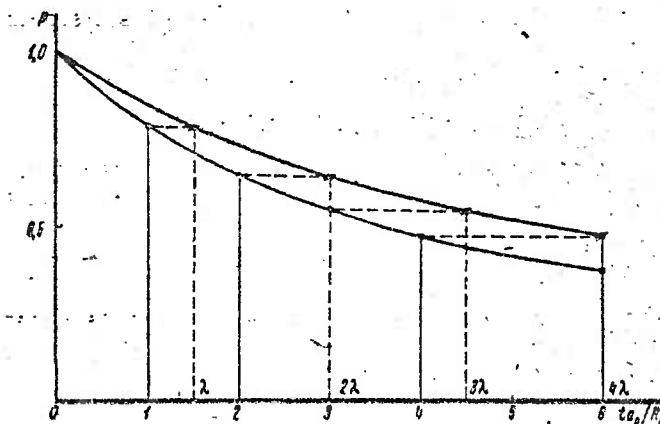
Fig. 38. Variation of the pattern pressure for a similarity modulus λ.

The comparable points will then be determined from the following

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relations:

$$\frac{r_1}{\sqrt[3]{E_1}} = \frac{r_2}{\sqrt[3]{E_2}},$$ (2.23)

where E_1 and E_2 are the energies released in the explosion of the two different charges.

The same ratio is obtained also for the comparable instants of time

$$\frac{t_1}{t_2} = \frac{\sqrt[3]{E_1}}{\sqrt[3]{E_2}}.$$ (2.24)

On the basis of the foregoing, the pressure field due to the explosion can be represented as a function of two parameters

$$p = f\left(\frac{t}{\sqrt[3]{E}}, \frac{r}{\sqrt[3]{E}}\right).$$ (2.25)

The explosion energy is proportional to the weight of the charge G. The latter, in turn, determines uniquely the radius of the equivalent spherical charge, since

$$G = \frac{4}{3}\pi R_{03}^3 \gamma,$$ (2.26)

where γ is the density of the explosive.

Putting $\gamma = 1.6$ g/cm^3, we obtain for TNT

$$R_{03} = 0.053\sqrt[3]{G}.$$ (2.27)

In Formula (2.27) the radius of the charge is in meters and its weight in kilograms.

We thus obtain in lieu of (2.25)

$$p = f\left(\frac{t}{R_{03}}, \frac{r}{R_{03}}\right).$$ (2.28)

In order to deal with a function of dimensionless parameters, it is best to rewrite Formula (2.28) in the form

$$p = p^* f\left(\frac{ta_0}{R_{03}}, \frac{r}{R_{03}}\right).$$ (2.29)

Obviously, for the maximum pressure, which occurs on the front of

the shock wave, the following equality holds true:

$$p = p^* \varphi \left(\frac{r}{R_{03}} \right).$$ (2.30)

One of the most important characteristics of an explosion, which in many cases determines its force effect on a partition, is the impulse of the pressures, defined by the integral

$$I = \int_0^t p \, dt,$$ (2.31)

where the time is reckoned from the instant when the shock wave front arrives at the given point.

In the last relation, substituting the value of the pressure by Formula (2.29) and going over to dimensionless time, we will have

$$I = \int_0^{\tau_\kappa} p^* f \left(\frac{t a_0}{R_{03}}, \; \frac{r}{R_{03}} \right) \frac{R_{03}}{a_0} \, d \left(\frac{t a_0}{R_{03}} \right) =$$

$$= p^* \frac{R_{03}}{a_0} \int_0^{\tau_\kappa} f \left(\frac{t a_0}{R_{03}}, \; \frac{r}{R_{03}} \right) d \left(\frac{t a_0}{R_{03}} \right),$$ (2.32)

where $\tau_k = t a_0 / R_{03}$ is the dimensionless time.

Since the integrand is a function of dimensionless parameters, it is obvious that at the same relative distance and for equal values of the relative time, the values of the impulses will be proportional to the radii of the charge or, what is the same, to the modulus of geometrical similarity λ.

Example. Establish the similarity parameters for an explosion, using the principles of dimensionality of the functions. The medium in which the explosion occurs has an initial density ρ_0 and an initial pressure p_0.

Solution. The main parameter characterizing the charge is the explosion energy E_0. The hydrodynamic fields, as shown above, are functions of the distance r and of the time t. Consequently, from the conditions of the problem, the process is determined by the following di-

mensional quantities:

$$E_0, \quad \rho_0, \quad p_0, \quad r, \quad t.$$

We shall assume the number of principal units to be equal to three (k = 3), — [T], [M], [L]. Then, on the basis of the π-theorem of similarity we can reduce the number of variables from the five mentioned above to two dimensionless criteria. To this end we write out first the dimensionalities of the initial parameters:

$$
\left.
\begin{aligned}
[p_0] &= ML^{-1}T^{-2}, \\
[\rho_0] &= ML^{-3}, \\
[t] &= T, \\
[r] &= L, \\
[E_0] &= ML^{\nu-1}T^{-2},
\end{aligned}
\right\}
\tag{2.33}
$$

where $\nu = 3$, 2, and 1, respectively, for motions with spherical, cylindrical, and plane symmetry. In writing down the dimension of energy we use the energy per unit length for motion with cylindrical symmetry and the energy per unit surface for motion with plane symmetry.

The dimensionless similarity criteria Π_1 and Π_2 will be sought in the form of fractions, the numerator of each is one of the n — k arbitrarily chosen quantities, and the denominator is made up of all the remaining mentioned quantities each raised to a power to be determined. Let, for example, the numerator of the first criterion be E_0 and that of the second p_0. Then

$$\Pi_1 = \frac{E_0}{\rho_0^{x_1} r^{x_2} t^{x_3}},$$

$$\Pi_2 = \frac{p_0}{\rho_0^{y_1} r^{y_2} t^{y_3}}.$$

In order to find the values of the exponents x_i and y_i, we equate the dimensionalities of the numerators and denominators of the expressions for Π_1 and Π_2:

$$[\Pi_1] = \frac{ML^{\nu-1}T^{-2}}{[ML^{-3}]^{x_1}L^{x_2}T^{x_3}}; \quad [\Pi_2] = \frac{ML^{-1}T^{-2}}{[ML^{-3}]^{y_1}L^{y_2}T^{y_3}},$$

we then obtain

$$x_1 = 1, \qquad y_1 = 1,$$

$$x_2 = \nu + 2, \qquad y_2 = 2,$$
$$x_3 = -2, \qquad y_3 = -2.$$

Thus,

$$\Pi_1 = \frac{E_0}{\rho_0 r^{\nu+2} t^{-2}} = \frac{E}{\rho_0} \frac{t^2}{r^{\nu+2}}; \tag{2.34}$$

$$\Pi_2 = \frac{p_0}{\rho_0 r^2 t^{-2}} = \frac{p_0 t^2}{\rho_0 r^2}. \tag{2.35}$$

The obtained similarity criteria fully determine the process under the initially assumed premises.

However, generally speaking, the second theorem of similarity theory admits of a nonunique choice of the defining criteria.

It is possible, for example, to stipulate in this case that the second of the similarity criteria depend only on one variable, for example $\underline{t}$.

Then, following the method already developed, we write

$$\Pi_2' = \frac{t}{p_0^{z_1} E_0^{z_2} \rho_0^{z_3}},$$

or, going over to the dimensionality formula

$$[\Pi_2'] = \frac{T}{[ML^{-1}T^{-2}]^{z_1} [ML^{\nu-1}T^{-2}]^{z_2} [ML^{-3}]^{z_3}},$$

hence

$$z_1 + z_2 + z_3 = 0, \qquad z_1 = -\frac{1}{2} - \frac{1}{\nu},$$
$$-z_1 + (\nu - 1) z_2 - 3z_3 = 0, \qquad z_2 = \frac{1}{\nu},$$
$$-2z_1 - 2z_2 = 1, \qquad z_3 = \frac{1}{2}.$$

Consequently,

$$\Pi_2' = \frac{t}{p_0^{-\frac{1}{2} - \frac{1}{\nu}} E_0^{\frac{1}{\nu}} \rho_0^{\frac{1}{2}}} = \frac{t p_0^{-\frac{1}{2} + \frac{1}{\nu}}}{E_0^{\frac{1}{\nu}} \rho_0^{\frac{1}{2}}}. \tag{2.36}$$

This form of the second similarity criterion was used by L. I. Sedov to study explosions with account of counterpressure.

§3. ESTIMATE OF HYDRODYNAMIC FIELDS IN THE CASE OF STRONG EXPLOSIONS IN AIR

Assuming that the main parameter characterizing the charge is the explosion energy E_0, and that the properties of the medium are determined sufficiently fully by the initial density ρ_0, the initial pressure p_0, and the adiabatic expansion coefficient k, L. I. Sedov indicated the following system of fundamental quantities for the estimate of the adiabatic perturbations of gas motion: E_0, ρ_0, p_0, k, r, t.

In accordance with the π-theorem of similarity theory, we can set up from these quantities the following three dimensionless combinations:*

$$k; \quad \lambda = \frac{E_0}{\rho_0} \cdot \frac{t^2}{r^{2+\nu}}; \tag{2.34}$$

$$\frac{p_0^{\frac{1}{2}+\frac{1}{\nu}} r}{E_0^{\frac{1}{\nu}} \rho_0^{\frac{1}{2}}} = \tau. \tag{2.36}$$

The effect of the initial pressure p_0, and consequently also of the parameter τ, is due only to the conditions of dynamic compatibility on the front of the shock wave. But if we study a strong explosion, then in the near zone the pressures on the front are two or three orders of magnitude higher than the initial pressure. Consequently, in such a problem one can neglect the initial pressure and the variable τ. Then the motion will be determined only by the dimensionless parameter λ, from the structure of which it follows that the process can be regarded as self-similar.

Assuming that $p_0/p_f \ll 1$, we obtain from (1.186)

$$\rho_\varphi \cong \frac{k+1}{k-1} \rho_0. \tag{2.37}$$

Consequently,

$$v_\varphi = \frac{\frac{k+1}{k-1}\rho_0 - \rho_0}{\frac{k+1}{k-1}\rho_0} = \frac{2}{k+1}N; \tag{2.38}$$

$$p_\phi = \frac{2}{k+1} \rho_0 N^2. \tag{2.39}$$

From the constancy of the density on the front of the shock wave it follows simultaneously that the parameter λ on the front should also remain constant at $\lambda = \lambda^*$. The law of motion of the shock wave can in this case be readily established directly from the relation (2.34):

$$r_\phi = \left(\frac{E_0}{\rho_\phi \lambda^*}\right)^{\frac{1}{2+\nu}} t_\phi^{\frac{2}{2+\nu}}. \tag{2.40}$$

Differentiating (2.40) with respect to time, we obtain

$$N = \frac{dr_\phi}{dt} = \frac{2}{2+\nu}\left(\frac{E_0}{r_\phi \lambda^*}\right)^{\frac{1}{2+\nu}} t^{-\frac{\nu}{2+\nu}} = \frac{2}{2+\nu}\frac{r_\phi}{t_\phi}. \tag{2.41}$$

Relations (2.40) and (2.41) are in good agreement with the experimental data on the propagation of an aerial shock wave near a charge.

Using Condition (2.41), we can write the equations of dynamic compatibility in the form*

$$\rho_\phi = \frac{k+1}{k-1}\rho_0; \tag{2.42}$$

$$p_\phi = \frac{8}{(2+\nu)^2}\frac{\rho_0}{k+1}\left(\frac{r_\phi}{t_\phi}\right)^2; \tag{2.43}$$

$$v_\phi = \frac{4}{(k+1)(2+\nu)}\left(\frac{r_\phi}{t_\phi}\right). \tag{2.44}$$

We now introduce dimensionless functions of the velocity, density, and pressure, defined by the relations

$$v = \frac{r}{t} V(\lambda); \tag{2.45}$$

$$\rho = \rho_0 R(\lambda); \tag{2.46}$$

$$p = \rho_0 \frac{r^2}{t^2} P(\lambda). \tag{2.47}$$

From a comparison of (2.42)-(2.44) and (2.45)-(2.47) it follows that the functions V, R, and P assume the following values at the front of the shock wave

$$R_\phi = \frac{k+1}{k-1}; \tag{2.48}$$

$$\rho_v - \frac{8}{(2+v)(k+1)}; \tag{2.49}$$

$$V_3 = \frac{4}{(k+1)(2+v)}. \tag{2.50}$$

We have previously had a system of fundamental differential equations of one-dimensional unsteady adiabatic motion of an ideal gas in the following form:

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial r} + \frac{1}{\rho} \frac{\partial p}{\partial r} = 0; \tag{1.90}$$

$$\frac{\partial \rho}{\partial t} + \rho \frac{\partial v}{\partial r} + v \frac{\partial \rho}{\partial r} + (v-1) \frac{\rho v}{r} = 0; \tag{1.92}$$

$$\frac{\partial}{\partial t} \frac{p}{\rho^x} + v \frac{\partial}{\partial r} \frac{p}{\rho^x} = 0. \tag{1.94}$$

Recognizing that

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \lambda} \frac{\partial \lambda}{\partial t} = 2 \frac{E_0}{\rho_0} \frac{t}{r^{2+v}} \frac{d}{d\lambda} = 2 \frac{\lambda}{t} \frac{d}{d\lambda};$$

$$\frac{\partial}{\partial r} = \frac{d}{d\lambda} \frac{\partial \lambda}{\partial r} = -(2+v) \frac{E_0}{\rho_0} \frac{t^2}{r^{3+v}} \left(\frac{d}{d\lambda} \right) = -(2+v) \frac{\lambda}{r} \frac{d}{d\lambda}.$$

we obtain after substituting the new variables in (1.90), (1.92), and (1.94)

$$\left. \begin{array}{l} \lambda \left[(mV-2) V' + m \frac{P'}{R} \right] = V^2 - V + 2 \frac{P}{R}, \\[2mm] \lambda \left[mV' + (mV-2) \frac{R'}{R} \right] = vV, \\[2mm] \lambda (mV-2) \left[\frac{P'}{P} - k \frac{R'}{R} \right] = 2(V-1), \end{array} \right\} \tag{2.51}$$

where m = 2 + v.

Integration of the system of ordinary differential equations (2.51) with the conditions on the front of the shock wave chosen as the initial conditions makes it possible to determine the functions V, P, and R, after which an estimate of the fields of the hydrodynamic elements entails no difficulty whatever.

It is necessary, however, in this case to use numerical methods, since the system (2.51) cannot be integrated in closed form. L. I. Sedov proposed a very simple method which makes it possible to obtain the algebraic integral of the system (2.51) with the aid of considerations of similarity theory. To this end, one considers the variation of the

total energy in a certain volume Q, bounded by a spherical (cylindrical, plane) surface λ = const. The displacement velocity of the points of such a surface will be, in accord with (2.41), $N = 2r/(2 + \nu)t$.

Since in accordance with the conditions of the problem we neglect the initial pressure p_0, we neglect simultaneously the initial energy of the gas $\varepsilon = p/(k - 1)\rho$ compared with the energy which the gas acquires as a result of the explosion. Therefore the total energy of the gas inside the volume Q, bounded by the surface λ = const, should remain constant.

Let the coordinate characterizing the volume Q be equal to $\underline{r}$. Expanding in accordance with the law (2.41), the surface λ = const yields after a time $\underline{dt}$ a volume increment $4\pi r^2 N dt$, which contains gas with energy $4\pi r^2 N dt[(\rho v^2/2) + \rho\varepsilon]$.

This energy should equal the energy $4\pi r^2 v dt[(\rho v^2/2) + \rho\varepsilon]$ which is delivered by the gas flowing over a time $\underline{dt}$ through a sphere of radius $\underline{r}$, and the work of the pressure forces $4\pi r^2 p v dt$.

We thus have

$$4\pi r^2 N dt\left(\frac{\rho v^2}{2} + \rho\varepsilon\right) = 4\pi r^2 v dt\left(\frac{\rho v^2}{2} + \rho\varepsilon\right) + 4\pi r^2 p v dt,$$

or, what is the same,

$$N\left(\frac{\rho v^2}{2} + \frac{p}{k-1}\right) = v\left(\frac{\rho v^2}{2} + \frac{p}{k-1} + p\right);$$

$$\frac{2}{2+\nu}\frac{r}{t}\left(\frac{\rho v^2}{2} + \frac{p}{k-1}\right) = v\left(\frac{\rho v^2}{2} + \frac{k}{k-1}p\right).$$

Finally, replacing the quantities $\underline{v}$, ρ, and $\underline{p}$ in the last equation by V, P, and R using Formulas (2.45)-(2.47), we obtain ultimately

$$RV^2\left(V - \frac{2}{2+\nu}\right) + \frac{2}{k-1}P\left(kV - \frac{2}{2+\nu}\right) = 0. \qquad (2.52)$$

Relation (2.52) is the integral of the system (2.51).

Using this relation, it is possible to obtain in closed form the solutions for V, R, and P, given in L. I. Sedov's book "Methods of Sim-

ilarity and Dimensionality in Mechanics."

As already mentioned, the initial pressure p_0 can be neglected, considering only the near zone of the explosion. At large distances, the pressure behind the front drops and the influence of the pressure of the undisturbed air ahead of the front of the shock wave becomes more and more important.

An account of the counterpressure greatly complicates the problem, for the motion is no longer self-similar. It becomes necessary then to integrate a system of partial differential equations by using some numerical method. This was done in the work by D.Ye. Okhotsimskiy, I.L. Kondrasheva, Z.P. Vlasov, and R.K. Kazakov "Calculation of Point Explosion with Allowance for Counterpressure." These authors obtained not only the fields of pressures, densities, velocities of flow, and temperatures for an explosion in a homogeneous atmosphere, but also calculated such important parameters as the impulses of the pressures in the positive and negative phases, together with the limiting distribution of the hydrodynamic elements.

The calculations are in good agreement with experiment.

Thus, it can be regarded that at the present time the problem of an aerial explosion has been solved completely and exhaustively.

The situation is much less satisfactory in the theory of underwater explosions, to the exposition of whose main problems we now proceed.

§4. L.I. SEDOV'S METHOD AS APPLIED TO THE STUDY OF UNDERWATER EXPLOSIONS

Whereas in the study of strong explosions we can assume the velocity of sound in the unperturbed medium to be small compared with the velocity of displacement of the shock wave over a sufficiently wide range of distances ($a_0/N < 0.1$) and we can neglect the counterpressure p_0, neither can be done in the study of underwater explosions, if it

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is recognized that in the latter case the role of "counterpressure" is played by the constant $c = 5400$ atm, and the shock-wave velocities can be only 3-5 times greater than the velocity of sound. Consequently, the direct use of L. I. Sedov's solution for the analysis of the propagation of underwater shock waves is impossible.

Nonetheless, the self-similarity principle proposed by Sedov can be used as a basis in the present case, too.

For simplicity we shall investigate isentropic motion, which is perfectly permissible for the case of an underwater explosion of ordinary explosives.

We write down the system of equations in terms of the variables v and a:

$$\left. \begin{array}{l} \frac{\partial v}{\partial t} + v \frac{\partial v}{\partial r} + \frac{2}{\varkappa - 1} a \frac{\partial a}{\partial r} = 0, \\ \frac{\partial a}{\partial t} + v \frac{\partial a}{\partial r} + \frac{\varkappa - 1}{2} a \frac{\partial v}{\partial r} + \frac{\varkappa - 1}{2} (\nu - 1) \frac{av}{r} = 0. \end{array} \right\} \qquad (2.53)$$

We put

$$\left. \begin{array}{l} v = \frac{r}{t} V(\lambda), \\ a = \frac{r}{t} A(\lambda), \end{array} \right\} \qquad (2.54)$$

where just as before

$$\lambda = \frac{E}{\rho_0} \frac{t^2}{r^m}. \qquad (2.35)$$

In the new variables, Eqs. (2.53) are rewritten in the form

$$\left. \begin{array}{l} \lambda \left\{ 2V' - mVV' - \frac{2m}{\varkappa - 1} A'A \right\} = -V^2 + V - \frac{2}{\varkappa - 1} A^2, \\ \lambda \left\{ 2A' - mVA' - \frac{\varkappa - 1}{2} mAV' \right\} = \\ = A - AV \frac{\varkappa + 1 + (\nu - 1)(\varkappa - 1)}{2}. \end{array} \right\} \qquad (2.55)$$

Eliminating the variable λ from the system (2.55), we arrive at one equation, where A is the sought function of V:

$$\lambda \left\{ A^2Vm\nu - mV^3 - A^2 \frac{2m - 4}{\varkappa - 1} + (2 + m) V^2 - 2V \right\} =$$
$$= mA^3 - AV^2m \left[1 + \frac{(\nu - 1)(\varkappa - 1)}{2} \right] + AV \left[(\varkappa + 1) + \right.$$
$$\left. + (\nu - 1)(\varkappa - 1) + m \frac{3 - \varkappa}{2} \right] - 2A. \qquad (2.56)$$

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Assume that at some instant of time t_0 (for example, at the instant that the detonation wave emerges on the surface of the charge) we have $r = r_0$, $v = v*$, $a = a*$, $N = N*$. Then on the basis of (2.54) we have at the same instant

$$V(\lambda*) = \frac{v*}{N*},$$
$$A(\lambda*) = \frac{a*}{N*}. \qquad (2.56a)$$

Knowing the quantities $V(\lambda*)$ and $A(\lambda*)$, the integral of the equation (2.56) can be obtained by using numerical methods.

Let us assume that as a result we have obtained a certain solution $A = A(V)$. Then, using the first equation of (2.55), we obtain

$$\lambda \frac{dV}{d\lambda} \left\{ 2 - mV - \frac{2m}{\varkappa - 1} \frac{dA}{dV} A(V) \right\} = V - V^2 - \frac{2}{\varkappa - 1} A^2(V),$$

so that the function λ is determined by the quadrature

$$\ln \frac{\lambda}{\lambda*} = \int_{V*}^{V} \frac{2 - mV - \frac{2m}{\varkappa - 1} \frac{dA}{dV} A(V)}{V - V^2 - \frac{2}{\varkappa - 1} A^2(V)} dV, \qquad (2.57)$$

and, consequently, we obtain the functions

$$V = V\left(\frac{\lambda}{\lambda*}\right) \quad \text{and} \quad A = A\left(\frac{\lambda}{\lambda*}\right).$$

The position of the front of the shock wave is determined by the conditions of dynamic compatibility

$$v_\phi = b(N - a_0);$$
$$N = a_0 + \frac{1}{b} v_\phi = a_0 + \frac{1}{b} \frac{r_\phi}{t} V(\lambda_\phi), \qquad (2.58)$$

but

$$N = \frac{dr_\phi}{dt}; \quad \lambda_\phi = \lambda(r_\phi, t).$$

We thus obtain the ordinary differential equation

$$\frac{dr_\phi}{dt} = a_0 + \frac{1}{b} \frac{r_\phi}{t} V(r_\phi, t), \qquad (2.59)$$

which when solved yields $r_f = r_f(t)$ and $\lambda_f = \lambda_f(t)$, thus completely solving the problem under the assumptions made.

Let us dwell in somewhat greater detail on motion with spherical symmetry. In this case Eq. (2.56), after substituting the numerical values of the coefficients $m = 5$, $\nu = 3$, $\kappa = 5.53$, is rewritten as

$$A = \frac{5A^3 - 27{,}65AV^2 + 9{,}265AV - 2A}{15A^2V - 5V^3 - 1{,}324A^2 + 7V^2 - 2V}.$$ (2.60)

Let us investigate the character of the integral curves in the half plane $V > 0$.

For this purpose it is necessary first of all to find the singular points of the differential equation (2.60).

They will be determined by the coefficients of an equation in the form

$$\frac{dy}{dx} = \frac{ax + by}{cx + dy}.$$

The course of the integral curves in the vicinity of the singular point depends in this case on the roots λ of the equation

$$\lambda^2 - (b + c)\lambda + bc - ad = 0.$$

If the roots λ_1 and λ_2 are real and have the same sign, the singular point is called a node. An infinite set of integral curves passes through a node (Fig. 39a).

If λ_1 and λ_2 are real and of opposite sign, only two curves pass through the singular point. The remaining curves do not pass through this point. A singular point of this type is called a saddle (Fig. 39b).

If the roots λ_1 and λ_2 are complex conjugates, then the integral curves in the vicinity of the singular point are a family of logarithmic spirals. The singular point is called a focus (Fig. 39c). If the roots λ_1 and λ_2 are purely imaginary, then the family of integral curves is a family of closed curves, surrounding the singular point. Such a point is called a center (Fig. 39d).

It may turn out that the roots λ_1 and λ_2 are real and equal to each other. Then, if $a = d$ and $b = c = \lambda$, all the integral curves pass

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through the singular point with a definite direction of the tangents. Such a point is called a dicritical node (Fig. 39f). There is in addition a node with one common tangent to the integral curves (Fig. 39e).
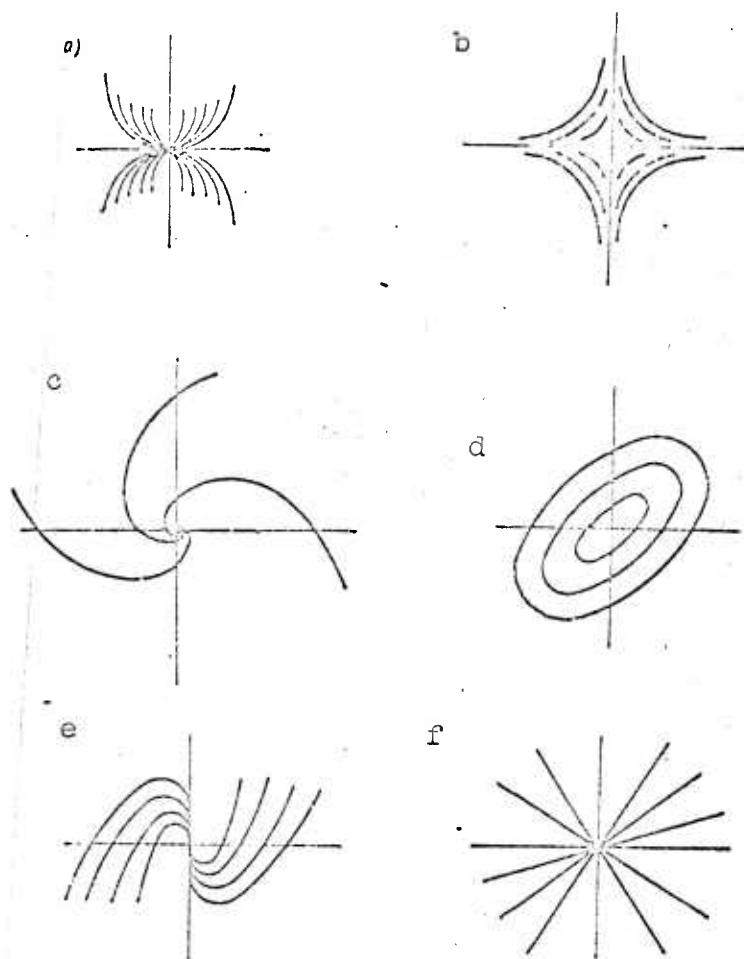
Fig. 39. Character of singular points of an ordinary differential equation.

Let us return to Eq. (2.60). In this case the singular points will be $\alpha(0, 0)$; $\beta(0, 2/5)$; $\gamma(0, 1)$; $\delta(0.5, 0.129)$; $\varepsilon(\infty, \infty)$.

At the points of the parabola $A^2 = V^2\kappa - [(\kappa + 13)V/10] + 2/5$ the integral curves have horizontal tangents; at the points determined by the equation

$$A^2 = \frac{5V^3 - 7V^2 + 2V}{15V - \frac{6}{\kappa - 1}},$$

(2.61)

and the tangents to the integral curves are vertical.*

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The possible course of the integral curves is shown in Fig. 40.

On the basis of the data of Table 7, calculating the pressure fields for the explosion of a TNT charge, we can assume* $V(\lambda^*) = 0.390$, $A(\lambda^*) = 1.104$.

The functions $V(\lambda/\lambda^*)$ and $A(\lambda/\lambda^*)$, calculated by numerical integration, are shown in Fig. 41.

TABLE 7

Координаты особой точки	Вид уравнения в окрестности особой точки	Характер особой точки	Угловые коэффициенты касательных к интегральным кривым
$A = 0$ $V = 0$	$A = \dfrac{A}{V}$	Дикритический узел	Произвольны
$V = \dfrac{2}{5}$ $A = 0$	$A = -2{,}565$	Седловина	Через особую точку проходят прямые $A = 0,\ V = -\dfrac{2}{5}$
$A = 0$ $V = 1$	$A = 6{,}80\dfrac{A}{V}$	Узел	$k_1 = 0$ $k_2 = \infty$
$A = 0{,}50$ $V = 0{,}129$	$A = \dfrac{1{,}065V + 2{,}485A}{1{,}621V + 0{,}611A}$	Узел	$k_1 = 2{,}20$ $k_2 = -0{,}48$

1) Coordinates of the singular point; 2) form of the equation in the vicinity of the singular point; 3) character of singular point; 4) slopes of the tangents to the integral curves; 5) dicritical node; 6) arbitrary; 7) saddle; 8) the lines A = = 0, V = −2/5 pass through the singular point; 9) node.

To stay in the region where the solution is unique, it is necessary to confine oneself to the range of variation $0 < \lambda < 6.1\lambda^*$.

What is striking is that the parameter λ vanishes at a value of V not equal to zero. But then it follows from the conditions of dynamic compatibility that the velocity of the shock wave at infinity differs greatly from the velocity of sound in the unperturbed medium. Equally essential differences from the parameters of the unperturbed liquid will be exhibited also by the other hydrodynamic elements.
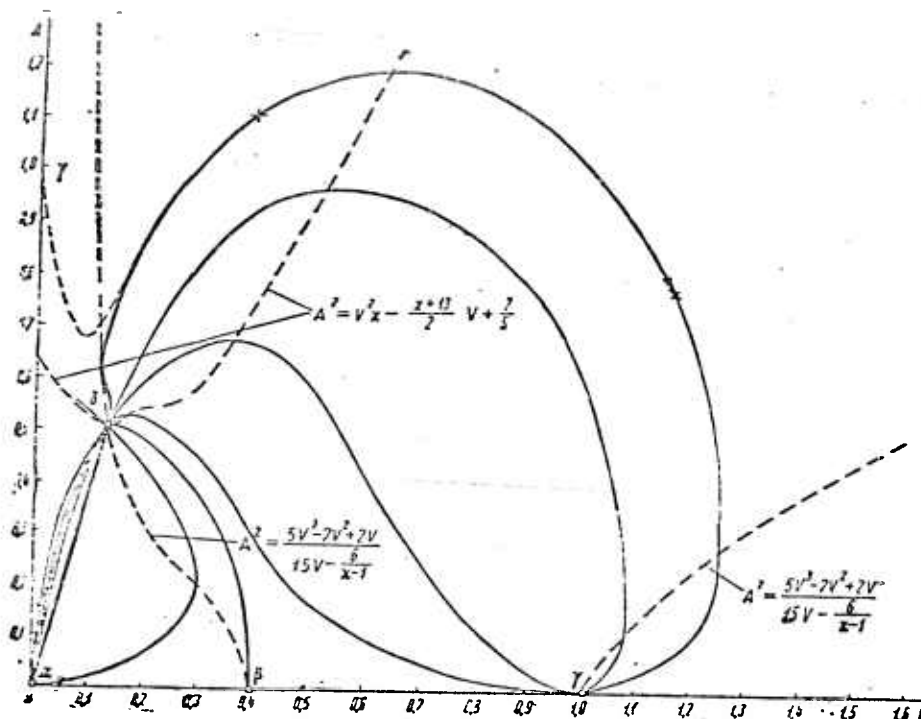
Fig. 40. Possible course of integral curves A =
= A(V) in the half plane A > 0. o) Singular point;
+) initial point; ×) point of maximum of the param-
eter λ: $\lambda = 6.1$. The dashed lines show parabolas
on which the tangents to the integral curves are
either vertical or horizontal.
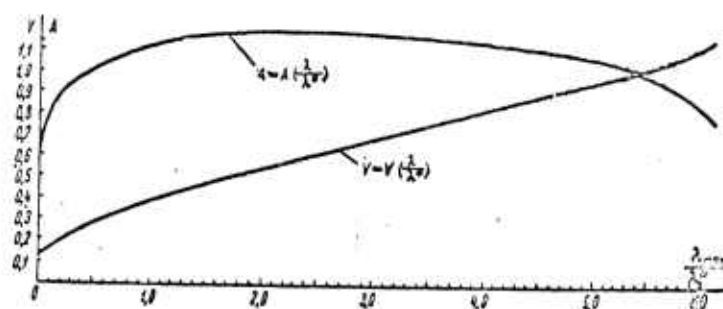
Fig. 41. The functions $A = A(\lambda/\lambda*)$ $V =$
$= V(\lambda/\lambda*)$.

This result, which does not correspond to reality, indicates that
the assumption that the energy E, being a defining parameter, can be
used to construct a solution only in a definite range of values, char-
acteristic of the region of relatively small distances from the center
of the explosion.

As regards the remaining range, it is necessary to assume there a

different defining dimensional parameter. Such a parameter may be, for example, the characteristic velocity of sound a*.

Putting

$$\eta = a^{*'} \frac{t^2}{r^2} \qquad (2.62)$$

and assuming

$$V = \frac{r}{t} V(\eta),$$

$$a = \frac{r}{t} A(\eta), \qquad (2.63)$$

we obtain in perfect analogy with the preceding case a fundamental differential equation relating the functions A and V, in the form

$$\dot{A} = \frac{A^3 - AV^2\varkappa + AV(\varkappa + 1) - A}{3A^2V - V^3 + 2V^2 - V} . \qquad (2.64)$$

The singular points of Eq. (2.64) will be A(0.0), B(0.1), C(1.0), D(0.50; 0.129). The points A, B, and C are nodes. Unlike the case previously considered, D will be a saddle in this case.

This makes it possible to couple together the two solutions in such a way that by continuously following the integral curve A(V) a transition is made from the strong perturbations in the liquid to acoustic perturbations [the singular point C(1.0) of Eq. (2.64) corresponds to a weak-discontinuity surface].

Starting from the conditions for the uniqueness of the solution, we can show that the coordinates that must be assumed for the contiguity point are V = 0.16; A = 0.78.

The value of the parameter η is determined, in analogy with the preceding, by the quadrature

$$\ln \frac{\eta}{\eta^*} = \int_{V_*}^{V} \frac{2 - 2V - \frac{4}{\varkappa - 1} A\dot{A}}{V - V^2 - \frac{2}{\varkappa - 1} A^2} \, dV. \qquad (2.65)$$

Since the integrand is negative in the considered range of variation, it is obvious that the parameter η increases as V changes from

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V* to zero, and becomes infinite at the singular point C(1, 0).

By way of the characteristic velocity of sound it is convenient to choose the velocity of sound on the front of the shock wave. Here, however, in view of the fact that as the shock wave moves the parameter η for the front of the wave decreases, one can no longer use a relation of the type (2.58) for an estimate of the shock-wave coordinates. It is necessary for this purpose to write down the dynamic compatibility conditions in a somewhat different form. It can be shown that for a wide range of variation of the parameters there exists a linear relation between the velocity of shock-wave front displacement and the velocity of sound on the front

$$a_\phi - a_0 = c(N - a_0),$$ (2.66)

from which it follows that

$$\frac{a_\phi - a_0}{v_\phi} = \frac{c}{b}$$ (2.67)
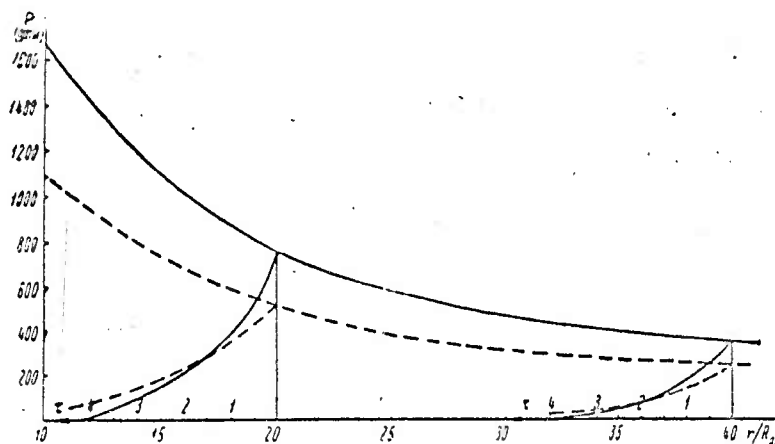
Fig. 42. Comparison of the values calculated by the method of L. I. Sedov (continuous line) and by the empirical formula of R. Koul (dashed).

or

$$\frac{\frac{r_\phi}{t} A - a_0}{\frac{r_\phi}{t} V} = \frac{c}{b},$$

and, consequently,

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$$\frac{r_\Phi}{t} = \frac{a_0}{A - \frac{c}{b} V}.$$ (2.68)

Figure 42 shows a comparison of the pressure curves calculated by the method presented here using the empirical formula of R. Koul.* It can be readily noted that for distances larger than 20 radii of the charge, the values of the pressure and the character of its variation in the compression zone are close to each other (Fig. 42).

§5. PRESSURE FIELDS AT LARGE DISTANCES FROM THE CENTER OF THE EXPLO-
SION. THE SOLUTION OF S.A. KHRISTIANOVICH

It was mentioned in the preceding section that the energy can be assumed as one of the main defining parameters only in a study of the near zone of an underwater explosion. At considerable distances from the center of the explosion the self-similar solutions must be sought by assuming that the motion is determined by the ratio of the distance to the time. Such an approach to the problem was used by S.A. Khris-tianovich. The gist of the solution obtained by him consists in the following.

The equations of one-dimensional unsteady isentropic motion of a liquid have the form**

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial r} + \frac{2}{n-1} a \frac{\partial a}{\partial r} = 0,$$ (2.69)

$$\frac{2}{n-1} \frac{\partial a}{\partial t} + a \frac{\partial v}{\partial r} + \frac{2}{n-1} v \frac{\partial a}{\partial r} + \frac{(\nu - 1) a v}{r} = 0.$$ (2.70)

Considering small pressures, we assume that

$$p = B\left[\left(\frac{\rho}{\rho_0}\right)^n - 1\right].$$

We then have

$$a_0 = \sqrt{\frac{nB}{\rho_0}},$$

$$a = a_0\left[1 + \frac{n-1}{2} \frac{p}{Bn}\right].$$ (1.206)

In the main system of equations we change over to new variables, putting

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$$\beta = \frac{r}{t}; \qquad (2.71)$$

$$\tau = \ln t. \qquad (2.72)$$

Here, obviously, $r = \beta e^{\tau}$

$$\frac{\partial v}{\partial \tau} = \frac{\partial v}{\partial r}\frac{\partial r}{\partial \tau} + \frac{\partial v}{\partial t}\frac{\partial t}{\partial \tau} = \frac{\partial v}{\partial r}r + \frac{\partial v}{\partial t}t,$$

$$\frac{\partial v}{\partial \beta} = \frac{\partial v}{\partial r}\cdot\frac{\partial r}{\partial \beta} = \frac{\partial v}{\partial r}\frac{r}{\beta},$$

hence

$$\frac{\partial v}{\partial r} = \frac{\beta}{r}\frac{\partial v}{\partial \beta}; \qquad (2.73)$$

$$\frac{\partial v}{\partial t} = \frac{1}{t}\left(\frac{\partial v}{\partial \tau} - \beta\frac{\partial v}{\partial \beta}\right). \qquad (2.74)$$

In full analogy

$$\frac{\partial a}{\partial r} = \frac{\beta}{r}\frac{\partial a}{\partial \beta}; \qquad (2.75)$$

$$\frac{\partial a}{\partial t} = \frac{1}{t}\left(\frac{\partial a}{\partial \tau} - \beta\frac{\partial a}{\partial \beta}\right). \qquad (2.76)$$

Substituting these relations into the system (2.69)-(2.70), we have

$$\frac{1}{t}\left(\frac{\partial v}{\partial \tau} - \beta\frac{\partial v}{\partial \beta}\right) + v\frac{\beta}{r}\frac{\partial v}{\partial \beta} + \frac{2}{n-1}a\frac{\beta}{r}\frac{\partial a}{\partial \beta} = 0;$$

$$\frac{\partial v}{\partial \tau} + (v - \beta)\frac{\partial v}{\partial \beta} + \frac{2}{n-1}a\frac{\partial a}{\partial \beta} = 0; \qquad (2.77)$$

$$\frac{2}{n-1}\left[\frac{\partial a}{\partial \tau} + (v - \beta)\frac{\partial a}{\partial \beta}\right] + a\frac{\partial v}{\partial \beta} + \frac{(\nu-1)av}{\beta} = 0. \qquad (2.78)$$

In the case of small pressures we can put

$$\beta = a_0(1 + \delta); \qquad (2.79)$$
$$a = a_0(1 + \alpha); \qquad (2.80)$$
$$v = a_0 M. \qquad (2.81)$$

We then obtain in place of the system (2.77)-(2.78)

$$\frac{\partial M}{\partial \tau} + [M - 1 - \delta]\frac{\partial M}{\partial \delta} + \frac{2}{n-1}(1 + \alpha)\frac{\partial \alpha}{\partial \delta} = 0; \qquad (2.82)$$

$$\frac{2}{n-1}\frac{\partial \alpha}{\partial \tau} + \frac{2}{n-1}(M - 1 - \delta)\frac{\partial \alpha}{\partial \delta} + (1 + \alpha)\frac{\partial M}{\partial \delta} +$$

$$+ \frac{(\nu-1)(1+\alpha)}{(1+\delta)}M = 0. \qquad (2.83)$$

We consider solutions that do not depend on the variable τ. We then obtain from Eq. (2.82)

$$\frac{\partial \alpha}{\partial \delta} = -\frac{n-1}{2(1+\alpha)}(M - 1 - \delta)\frac{\partial M}{\partial \delta}. \qquad (2.84)$$

Substituting (2.84) in (2.83) we have

$$-\frac{(M-1-\delta)^2}{1+\alpha}\frac{\partial M}{\partial\delta} + (1+\alpha)\frac{\partial M}{\partial\delta} + \frac{(\nu-1)(1+\alpha)}{1+\delta}M = 0,$$

or

$$-(M^2+1+\delta^2-2M+2\delta-2M\delta)\frac{\partial M}{\partial\delta} +$$
$$+ (1+2\alpha+\alpha^2)\frac{\partial M}{\partial\delta} + \frac{(\nu-1)(1+2\alpha+\alpha^2)}{1+\delta}M = 0.$$

Discarding terms of second and higher order of smallness, we obtain ultimately

$$(M-\delta+\alpha)\frac{\partial M}{\partial\delta} + \frac{(\nu-1)M}{2} = 0. \qquad (2.85)$$

We shall consider first motions with spherical symmetry. The initial equations will be

$$\frac{\partial M}{\partial\delta} - \frac{2}{n-1}\frac{\partial\alpha}{\partial\delta} = 0; \qquad (2.86)$$

$$(M+\alpha-\delta)\frac{\partial M}{\partial\delta} + M = 0. \qquad (2.87)$$

For the initial data we assume that $\alpha = \alpha_0$ and $\delta = \delta_0$ for $M = 0$, and in accordance with (1.206)

$$\alpha_0 = \frac{n-1}{2}\frac{p_0}{Bn}.$$

Integrating (2.86), we obtain

$$\alpha - \alpha_0 = \frac{n-1}{2}M. \qquad (2.88)$$

Substituting (2.88) in (2.87) we get

$$\left(\alpha_0 + \frac{n-1}{2}M + M - \delta\right)\frac{\partial M}{\partial\delta} + M = 0;$$
$$\left(\alpha_0 - \delta + \frac{n+1}{2}M\right)\frac{\partial M}{\partial\delta} + M = 0;$$
$$\frac{\partial\delta}{\partial M} = -\frac{\alpha_0}{M} + \frac{\delta}{M} - \frac{(n+1)}{2},$$

or

$$\frac{\partial\delta}{\partial M} - \frac{\delta}{M} = -\frac{\alpha_0}{M} - \frac{(n+1)}{2}. \qquad (2.89)$$

Equation (2.89) is an ordinary first-order differential equation. Its general integral is

$$\delta - \alpha_1 = CM - \frac{(n+1)}{2} M \ln M, \tag{2.90}$$

where C is the integration constant.

For small pressures, in accordance with the dynamic compatibility conditions the velocity of displacement of the shock wave front is

$$N = \dot{a}_0 \left[1 + \frac{n+1}{4} \frac{p - p_0}{Bn} \right], \tag{2.91}$$

and the velocity of motion of the particles is determined by the expression

$$v = a_0 \frac{p - p_0}{Bn}. \tag{2.92}$$

Let us put $p_0 = 0$, and then Eq. (2.90) is rewritten in the form

$$\delta = CM - \frac{n+1}{2} M \ln M. \tag{2.93}$$

Since

$$\delta = \frac{\beta}{a_0} - 1 = \frac{r}{t a_0} - 1,$$

$$M = \frac{v}{a_0} = \frac{p}{Bn},$$

we have

$$\frac{r}{t} = a_0 \left(1 + C \frac{p}{Bn} - \frac{n+1}{2} \frac{p}{Bn} \ln \frac{p}{Bn} \right).$$

The arbitrary constant C is conveniently replaced by another arbitrary constant, putting

$$C = \frac{n+1}{2} \ln \frac{p^*}{Bn}.$$

Thus

$$r = a_0 \left(1 + \frac{n+1}{2} \frac{p}{Bn} \ln \frac{p^*}{p} \right) t. \tag{2.94}$$

Let us find the velocity of propagation N of the shock wave front. Differentiating (2.90) with respect to $\underline{t}$, we obtain

$$dr = a_0 \left(1 + \frac{n+1}{2} \frac{p}{Bn} \ln \frac{p^*}{p} \right) dt +$$
$$+ a_0 t \frac{n+1}{2Bn} \left(\ln \frac{p^*}{p} - 1 \right) dp. \tag{2.95}$$

But on the basis of (2.91) we have

$$dr = a_0 \left[1 + \frac{n+1}{4Bn} p \right] dt.$$

(2.96)

Substituting (2.96) in (2.95) we obtain after grouping like terms

$$\frac{1}{2} p \, dt = p \ln \frac{p^*}{p} \, dt + t \left[\ln \frac{p^*}{p} - 1 \right] dp,$$

(2.97)

or

$$\frac{dt}{t} = \frac{\ln \frac{p^*}{p} - 1}{p \left(\frac{1}{2} - \ln \frac{p^*}{p} \right)} \, dp = - \left[1 - \frac{1}{2 \left(\ln \frac{p^*}{p} - \frac{1}{2} \right)} \right] d \ln p.$$

Integrating, we get

$$\ln t = - \ln p + \frac{1}{2} \int \frac{d \ln p}{\ln \frac{p^*}{p} - \frac{1}{2}} =$$

$$= - \ln p - \frac{1}{2} \int \frac{d \ln \frac{p^*}{p}}{\ln \frac{p^*}{p} - \frac{1}{2}} =$$

$$= \ln C_1 - \ln p - \frac{1}{2} \ln \left[\ln \frac{p^*}{p} - \frac{1}{2} \right],$$

hence

$$t = \frac{C}{p \sqrt{\ln \frac{p^*}{p} - \frac{1}{2}}}.$$

(2.98)

Taking (2.98) into account, Expression (2.94) can be written in the form

$$\frac{r}{R_0} = \frac{A}{p} \frac{1 + \frac{1}{2} (n+1) \frac{p}{Bn} \ln \frac{p^*}{p}}{\sqrt{\ln \frac{p^*}{p} - 0,5}},$$

(2.99)

where R_0 is the radius of the charge and A is an arbitrary constant.

Equation (2.99) enables us to estimate the value of the pressure on the front of the wave at a distance $\underline{r}$ from the center of the explosion, if we know the values of the constants A and p^*.

Let us now calculate the time of action of the shock wave.

In accordance with (2.71) and (2.79) we obtain

$$t = \frac{r_m - r}{a_0} = t_m (\delta_m - \delta),$$

(2.100)

where

$$\delta = \frac{n+1}{2} \frac{p^*}{Bn} \ln \frac{p^*}{p};$$

(2.101)

$$\delta_m = \frac{n+1}{2} \frac{p_m}{Bn} \ln \frac{p^*}{p_m};$$

(2.102)

$$t_m = \frac{AR_0}{a_0} \frac{1}{p_m} \frac{1}{\sqrt{\ln \frac{p^*}{p_m} - 0,5}},$$

(2.103)

or, after substituting (2.101)-(2.103) in (2.100),

$$t = \frac{n+1}{2} \frac{A}{Bn} \frac{R_0}{a_0} \frac{\ln\left(\frac{p^*}{p_m}\right) - \frac{p}{p_m} \ln \frac{p^*}{p}}{\sqrt{\ln \frac{p^*}{p_m} - 0,5}}.$$

(2.104)

Usually the time variation of pressure in a shock wave is characterized by the exponential relation

$$p = p_m e^{-\frac{t}{\theta}},$$

(2.105)

from which it follows that the time constant θ corresponds to the time $\underline{t}$ during which the pressure drops by a factor $\underline{e}$.

Thus, on the basis of (2.105) we obtain from (2.104)

$$\frac{\theta a_0}{R_0} = \frac{n+1}{2} \frac{A}{Bn} \frac{(e-1) \ln\left(\frac{p^*}{p_m}\right) - 1}{e \sqrt{\ln \frac{p^*}{p_m} - 0,5}}.$$

(2.106)

For TNT we have

$$R_0 = 0,053 \sqrt[3]{G},$$

where G is the weight of the charge in kg and R_0 is the radius in meters.

If in Formulas (2.99) (2.106) we set p* = 17,000 atm and A = 16,200, then these formulas yield values of frontal pressure p_m and time constant θ that are close to experiment for p < 250 atm.

We have

$$\bar{r} = \frac{16\,200}{p} \frac{1 + 1,87 \cdot 10^{-4} p \ln \frac{17\,000}{p}}{\sqrt{\ln \frac{17\,000}{p} - 0,5}};$$

(2.107)

$$\frac{\theta a_0}{r_0} = 1,11 \frac{1,73 \ln \frac{17\,000}{p_m} - 1}{\sqrt{\ln \frac{17\,000}{p_m} - 0,5}}.$$

(2.108)
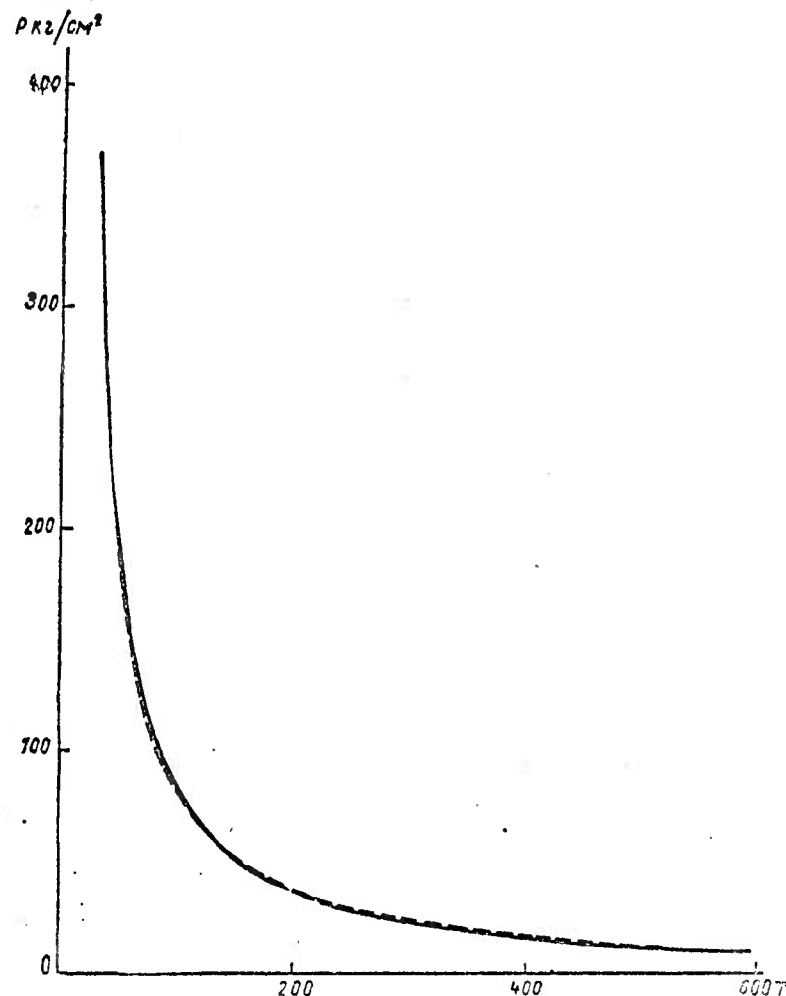
Fig. 43. Comparison of the results of the theory (solid line) with the experimental data (dashed line) on the values of the pressure on the front of a shock wave.

Comparison of the results of calculations by means of Formulas (2.107) and (2.108) with the empirical data is given in Table 8 and shown on Figs. 43 and 44.

At large distances from the center of the explosion, Relation (2.107) gives the asymptotic law for the decrease of pressure with increasing distance, as obtained by L. D. Landau.*

Motions with cylindrical symmetry can be investigated in perfect analogy with the foregoing.

The initial equations will be

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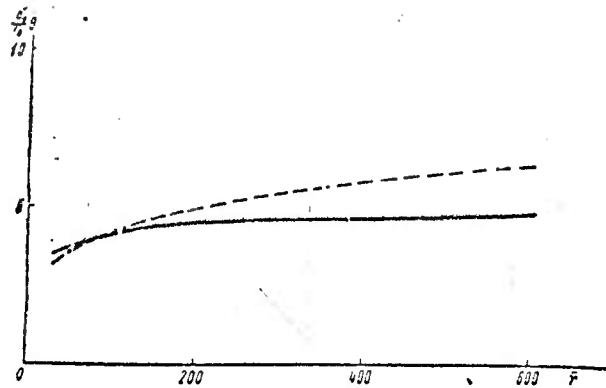
Fig. 44. Comparison of the results of the theory (solid line) and the experimental data (dashed line) on the values of the time constant.

$$\frac{\partial M}{\partial \delta} - \frac{2}{n-1} \frac{\partial a}{\partial \delta} = 0; \qquad (2.109)$$

$$(M + a - \delta) \frac{\partial M}{\partial \delta} + \frac{M}{2} = 0. \qquad (2.110)$$

TABLE 8

$\bar{r}$	1 Расчет		По эмпирическим формулам 2 Р. Коула	
	3 p_m, $кг/см^2$	$\frac{a_0}{r_0}\theta$	3 p_m, $кг/см^2$	$\frac{a_0}{R_0}\theta$
30	364	3,5	315	3,3
40	250	3,6	228	3,5
50	188	3,8	177	3,8
60	150	3,9	144	3,9
90	91,0	4,1	91,0	4,2
120	65,0	4,2	65,7	4,6
180	40,5	4,1	41,6	4,8
240	29,2	4,6	30,0	5,1

1) Calculation; 2) from the empirical formulas of R. Koul; 3) p_m, kg/cm^2.

Taking the same initial data as before, we arrive in the integration of the system (2.109)-(2.110), after elementary transformations, at the ordinary differential equation

$$\frac{\partial \delta}{\partial M} - \frac{2\delta}{M} = -\frac{2\alpha_1}{M} - (n+1). \qquad (2.111)$$

Its general solution is

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$$\delta - a_1 = (n+1)M + C_1 M^2, \qquad (2.112)$$

where C_1 is the constant of integration.

Assuming as before that p_0 and consequently α_0 vanish and recalling that

$$\delta = \frac{\beta}{a_0} - 1 = \frac{r}{ta_0} - 1,$$

we obtain

$$\frac{r}{t} = a_0 [1 + (n+1)M + C_1 M^2],$$

but

$$M = \frac{v}{a_0} = \frac{p}{Bn}.$$

Consequently,

$$\frac{r}{t} = a_0 \left[1 + (n+1)\frac{p}{Bn} + C_1 \frac{p^2}{B^2 n^2} \right],$$

or, introducing a new integration constant p*,

$$\frac{r}{t} = a_0 \left[1 + (n+1)\frac{p}{Bn} \left(1 - \frac{p}{p^*}\right) \right]. \qquad (2.113)$$

Differentiating, we have

$$dr = a_0 \left[1 + (n+1)\frac{p}{Bn} \left(1 - \frac{p}{p^*}\right) \right] dt +$$
$$+ a_0 t \frac{n+1}{Bn} \left[1 - 2\frac{p}{p^*} \right] dp. \qquad (2.114)$$

But in accordance with (2.96)

$$dr = a_0 \left[1 + \frac{n+1}{4Bn} p \right] dt.$$

Thus, after substituting (2.96) in (2.114) we have for the front of the shock wave the equation

$$\frac{1}{4} p \, dt = p \left(1 - \frac{p}{p^*}\right) dt + t \left(1 - 2\frac{p}{p^*}\right) dp,$$

or

$$\frac{dt}{t} = \frac{1 - 2\frac{p}{p^*}}{-\frac{3}{4} p + \frac{p^2}{p^*}} \, dp.$$

Integrating it, we obtain

$$t = \frac{D_1 R_0}{\left(\frac{p}{p^*}\right)^{2/3} \left[1 - \frac{4}{3}\frac{p}{p^*}\right]^{3/4}}. \qquad (2.115)$$

Substituting this result in (2.113), we obtain the connection between the pressure on the front of the shock wave and the relative distance in the form

$$\bar{r} = D_2 \frac{1 + (n+1)\frac{p}{Bn}\left(1 - \frac{p}{p^*}\right)}{\left(\frac{p}{p^*}\right)^{2/3}\left[1 - \frac{4}{3}\frac{p}{p^*}\right]^{3/4}}. \qquad (2.116)$$

The arbitrary constants D_2 and p* can be determined from the value of the pressure on the front of the shock wave and the time of action at some one distance from the center of the explosion.

§6. ESTIMATE OF HYDRODYNAMIC FIELDS IN THE ACOUSTIC APPROXIMATION

As is well known, the propagation of acoustic waves is characterized by the following: constant propagation of sound at all points of the medium; small variation of density compared with the density of the unperturbed medium: $\rho = \rho_0 + \rho'$ with $\rho'/\rho_0 \ll 1$, and also low particle velocity compared with the velocity of sound ($v/a_0 \ll 1$).

Satisfaction of these conditions imposes entirely different requirements on the variation of the pressure fields in water and in air where the methods of acoustics can be employed.

Actually, the velocity of sound in an ideal gas is given by the relation

$$a = \sqrt{\frac{kp}{\rho}} = \sqrt{\frac{kp_0}{\rho_0} \cdot \frac{1 + \frac{p'}{p_0}}{1 + \frac{\rho'}{\rho_0}}} = a_0 \sqrt{\frac{1 + \frac{p'}{p_0}}{1 + \frac{\rho'}{\rho_0}}}.$$

Considering an adiabatic process and using the equation $p/\rho^k = p_0/\rho_0^k$, we obtain with difficulty

$$a = a_0 \sqrt{\left(1 + \frac{\rho'}{\rho_0}\right)^{k-1}} = a_0 \left(1 + \frac{\rho'}{\rho_0}\right)^{\frac{k-1}{2}},$$

from which it follows that if the tolerable error in the estimate of

the velocity of sound is 5%, the value of ρ' must be assumed to be not more than 0.25 ρ_0, and the value of p' must be less than 0.35 p_0.

For water we have

$$a = \sqrt{\frac{n(p+B)}{\rho}} = \sqrt{\frac{n(p_0+B)}{\rho_0} \frac{1 + \frac{p'}{p_0+B}}{1 + \frac{\rho'}{\rho_0}}} =$$

$$= a_0 \sqrt{\frac{1 + \frac{p'}{p_0+B}}{1 + \frac{\rho'}{\rho_0}}} = a_0 \sqrt{\left(1 + \frac{\rho'}{\rho_0}\right)^{n-1}} =$$

$$= a_0 \left(1 + \frac{\rho'}{\rho_0}\right)^{\frac{n-1}{2}}.$$

Consequently, the error in the estimate of the velocity of sound will be 5% if $\rho'/\rho_0 \simeq 0.016$. Corresponding to this value is a pressure of 350 atm.

In view of this, the acoustic approximation can be successfully used for an underwater explosion.

Let us present the hydrodynamic equations in the acoustic approximation, which are needed for what is to follow.

We assume that

$$\rho = \rho_0 + \rho',$$
$$p = p_0 + p',$$

and then we obtain for one-dimensional motion

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial r} + \frac{1}{\rho_0 + \rho'} \frac{\partial p'}{\partial r} = 0; \tag{2.117}$$

$$\frac{\partial \rho'}{\partial t} + v \frac{\partial \rho'}{\partial r} + (\rho_0 + \rho') \frac{\partial v}{\partial r} + \frac{(\nu - 1)(\rho_0 + \rho')}{r} v = 0. \tag{2.118}$$

Neglecting in Eqs. (2.117) and (2.118) quantities of second order of smallness and taking into consideration the fact that

$$\frac{\partial \rho}{\partial t} = \frac{d\rho}{dp} \frac{\partial p}{\partial t} = \frac{1}{a_0^2} \frac{\partial p}{\partial t},$$

we have

$$\left. \begin{array}{l} \rho_0 \dfrac{\partial v}{\partial t} + \dfrac{\partial p'}{\partial r} = 0, \\[2mm] \dfrac{1}{a_0^2} \dfrac{\partial p'}{\partial t} + \rho_0 \dfrac{\partial v}{\partial r} + \dfrac{(\nu - 1)\rho_0 v}{r} = 0. \end{array} \right\} \tag{2.119}$$

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Differentiating the first equation of (2.119) with respect to r, the second with respect to t, and subtracting the first from the second, we obtain

$$\frac{\partial^2 p'}{\partial r^2} - \frac{1}{a_0^2} \frac{\partial^2 p'}{\partial t^2} - \frac{(\nu - 1)}{r} \rho_0 \frac{\partial v}{\partial t} = 0. \tag{2.120}$$

Finally, substituting $\rho_0 \frac{\partial v}{\partial t} = -\frac{\partial p'}{\partial r}$, in (2.120), we obtain

$$\frac{\partial^2 p'}{\partial r^2} - \frac{1}{a_0^2} \frac{\partial^2 p'}{\partial t^2} + \frac{(\nu - 1)}{r} \frac{\partial p'}{\partial r} = 0. \tag{2.121}$$

Equation (2.121) is called the wave equation. It is easy to show that under the assumptions formulated above, equations of this type will be satisfied also by the functions ρ' and v.

Let us consider some particular cases.

Let the motion have plane symmetry, and then

$$\frac{\partial^2 p'}{\partial r^2} - \frac{1}{a_0^2} \frac{\partial^2 p'}{\partial t^2} = 0. \tag{2.122}$$

We introduce the variables

$$\left. \begin{array}{l} \xi = t - \dfrac{r}{a}, \\[2mm] \eta = t + \dfrac{r}{a}. \end{array} \right\} \tag{2.123}$$

The derivatives of p' in the new variables will be

$$\frac{\partial p'}{\partial r} = \frac{\partial p'}{\partial \xi} \frac{\partial \xi}{\partial r} + \frac{\partial p'}{\partial \eta} \frac{\partial \eta}{\partial r};$$

$$\frac{\partial^2 p'}{\partial r^2} = \frac{\partial^2 p'}{\partial \xi^2} \left(\frac{\partial \xi}{\partial r} \right)^2 + 2 \frac{\partial^2 p'}{\partial \xi \partial \eta} \frac{\partial \xi}{\partial r} \frac{\partial \eta}{\partial r} +$$

$$+ \frac{\partial^2 p'}{\partial \eta^2} \left(\frac{\partial \eta}{\partial r} \right)^2 + \frac{\partial p'}{\partial \xi} \frac{\partial^2 \xi}{\partial r^2} + \frac{\partial p'}{\partial \eta} \cdot \frac{\partial^2 \eta}{\partial r^2};$$

$$\frac{\partial^2 p'}{\partial t^2} = \frac{\partial^2 p'}{\partial \xi^2} \left(\frac{\partial \xi}{\partial t} \right)^2 + 2 \frac{\partial^2 p'}{\partial \xi \partial \eta} \frac{\partial \xi}{\partial t} \frac{\partial \eta}{\partial t} +$$

$$+ \frac{\partial^2 p'}{\partial \eta^2} \left(\frac{\partial \eta}{\partial t} \right)^2 + \frac{\partial p'}{\partial \xi} \frac{\partial^2 \xi}{\partial t^2} + \frac{\partial p'}{\partial \eta} \frac{\partial^2 \eta}{\partial t^2}.$$

From (2.123) it follows that

$$\frac{\partial \xi}{\partial r} = -\frac{1}{a}; \quad \frac{\partial \eta}{\partial r} = \frac{1}{a};$$

$$\frac{\partial^2 \xi}{\partial r^2} = \frac{\partial^2 \eta}{\partial r^2} = 0; \quad \frac{\partial \xi}{\partial t} = \frac{\partial \eta}{\partial t} = 1;$$

$$\frac{\partial^2 \xi}{\partial t^2} = \frac{\partial^2 \eta}{\partial t^2} = 0.$$

On the basis of the relations written out, Eq. (2.122) assumes the following form in the new variables

$$\frac{\partial^2 p'}{\partial \xi^2} + 2\frac{\partial^2 p'}{\partial \xi \partial \eta} + \frac{\partial^2 p'}{\partial \eta^2} - a^2 \left(\frac{1}{a^2} \frac{\partial^2 p'}{\partial \xi^2} \right. -$$

$$\left. - 2\frac{1}{a^2} \frac{\partial^2 p}{\partial \xi \partial \eta} + \frac{1}{a^2} \frac{\partial^2 p}{\partial \eta^2} \right) = 0$$

or, after grouping similar terms,

$$\frac{\partial^2 p'}{\partial \xi \partial \eta} = 0. \tag{2.124}$$

Let us integrate first with respect to ξ and then with respect to η, and obtain

$$p' = f_1(\xi) + f_2(\eta), \tag{2.125}$$

or

$$p' = f_1 \left(t - \frac{r}{a_0} \right) + f_2 \left(t + \frac{r}{a_0} \right). \tag{2.126}$$

The solution (2.126) is a sum of two traveling waves.

If the wave propagates in the positive $\underline{r}$ direction, then $f_2 \equiv 0$, and if it propagates in the negative direction then $f_1 \equiv 0$.

The physical meaning of (2.126) lies in the fact that in the case of motion with plane symmetry any perturbation arising at some point $\underline{r}$ moves with velocity a_0 and remains unchanged in form.

Let us proceed to consider the wave equation for a motion with spherical symmetry.

In this case (2.121) assumes the form

$$\frac{\partial^2 p'}{\partial r^2} + \frac{2}{r} \frac{\partial p'}{\partial r} - \frac{1}{a_0^2} \frac{\partial^2 p'}{\partial t^2} = 0. \tag{2.127}$$

Let us put

$$p' = \frac{P(r, t)}{r}, \tag{2.128}$$

then

$$\frac{\partial p'}{\partial r} = \frac{1}{r} \frac{\partial P}{\partial r} - \frac{P}{r^2};$$

$$\frac{\partial^2 p'}{\partial r^2} = \frac{1}{r} \frac{\partial^2 P}{\partial r^2} - \frac{2}{r^2} \frac{\partial P}{\partial r} + 2 \frac{P}{r^3};$$

$$\frac{\partial^2 p'}{\partial t^2} = \frac{1}{r} \frac{\partial^2 P}{\partial t^2}.$$

Substituting the result obtained in (2.127), we get

$$\frac{1}{r} \frac{\partial^2 P}{\partial r^2} - \frac{2}{r^2} \frac{\partial P}{\partial r} + 2 \frac{P}{r^3} + \frac{2}{r^2} \frac{\partial P}{\partial r} - 2 \frac{P}{r^3} -$$

$$- \frac{1}{a^2} \frac{1}{r} \frac{\partial^2 P}{\partial t^2} = 0,$$

or

$$\frac{\partial^2 P}{\partial r^2} - \frac{1}{a_0^2} \frac{\partial^2 P}{\partial t^2} = 0. \tag{2.129}$$

In accordance with (2.126), the solution of (2.129) will be

$$P = f_1 \left(t - \frac{r}{a_0} \right) + f_2 \left(t + \frac{r}{a_0} \right).$$

Consequently,

$$p' = \frac{f_1 \left(t - \frac{r}{a_0} \right) + f_2 \left(t + \frac{r}{a_0} \right)}{r}. \tag{2.130}$$

The first term of the right half of Eq. (2.130) is a wave propagating radially from the center, while the second term is a wave converging toward the center.

Unlike the motion with plane symmetry, the amplitude of the spherical wave decreases in inverse proportion to the distance from the center.
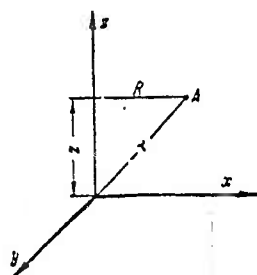
Fig. 45. System of cylindrical coordinates.
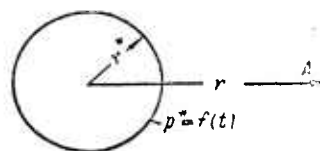
Fig. 46. Boundary condition of motion with spherical symmetry.

In the case of motion with cylindrical symmetry, Eq. (2.121) assumes the form

$$\frac{\partial^2 p'}{\partial r^2} + \frac{1}{r}\frac{\partial p'}{\partial r} - \frac{1}{a_0^2}\frac{\partial^2 p'}{\partial t^2} = 0. \tag{2.131}$$

A solution of this equation is

$$p' = \int_R^\infty \frac{f_1\left(t - \frac{r}{a_0}\right)}{\sqrt{r^2 - R^2}}\,dr + \int_R^\infty \frac{f_2\left(t + \frac{r}{a_0}\right)}{\sqrt{r^2 - R^2}}\,dr. \tag{2.132}$$

The notation used in Formula (2.132) is indicated in Fig. 45.

It is easy to apply the previously presented results to an analysis of the motion of a liquid due to an underwater explosion. Let us consider motion with spherical symmetry.

Assume that on some spherical surface there is specified the variation of the pressure as a function of the time, and let the estimate of the pressures in the entire space, starting from this surface, be possible in the acoustic approximation (Fig. 46).

Then for a wave diverging from the center we obviously obtain

$$p'(r, t) = p^* \frac{r^*}{r} f_1\left(t - \frac{r - r^*}{a_0}\right). \tag{2.133}$$

It is of interest also to estimate the values of the velocities of the motion of the liquid behind the front. Substituting (2.133) into the Euler equation of motion, we obtain

$$\frac{\partial v}{\partial t} = -\frac{1}{\rho_c}\frac{\partial p}{\partial r} = -\frac{1}{\rho_0}\frac{\partial}{\partial r}\left[p^* \frac{r^*}{r} f_1\left(t - \frac{r - r^*}{a_0}\right)\right] =$$
$$= \frac{1}{\rho_0 a_0}\frac{p^* r^*}{r} f_1'\left(t - \frac{r - r^*}{a_0}\right) + \frac{1}{\rho_0 r^2} p^* r^* f_1\left(t - \frac{r - r^*}{a_0}\right).$$

Integrating from the instant of arrival of the wave perturbation at the point, we obtain

$$v = \frac{1}{\rho_0 a_0}\int_0^t p^* \frac{r^*}{r} f_1'\left(t - \frac{r - r^*}{a_0}\right)dt +$$
$$+ \frac{1}{\rho_0 r}\int_0^t p^* \frac{r^*}{r} f_1\left(t - \frac{r - r^*}{a_0}\right)dt,$$

or on the basis of (2.133)

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$$v = \frac{1}{\rho_0 a_0}(p - p_0) + \frac{1}{\rho a} \int_0^t (p - p_0)\, dt. \tag{2.134}$$

From the last formula we see that the velocity of the liquid at the given point is a function not only of the pressure at the considered instant of time, but also of all the preceding changes in the pressure at the point, starting from the time when the perturbation reached it for the first time.

§7. APPROXIMATE ESTIMATE OF THE PRESSURE FIELDS IN THE CASE OF AN AERIAL EXPLOSION IN AN UNBOUNDED SPACE

Experimental research carried out on a rather broad basis on the shock waves occurring in an aerial explosion have confirmed the correctness of the laws of similarity theory, and have made it possible to obtain simple computation formulas that are convenient for practical use.

Using as the basis the general functional equation

$$\Delta p = f\left(\frac{\sqrt[3]{G}}{r}\right), \tag{2.135}$$

where G is the weight of the charge and $\underline{r}$ is the distance, M.A. Sadovskiy established on the basis of a thorough analysis of both the domestic and foreign experimental material that the value of the excess pressure on the front of the shock wave, in the case of an aerial explosion of a TNT charge, is determined by the relation*

$$\Delta p_\phi = 0.76\frac{\sqrt[3]{G}}{r} + 2.55\frac{\sqrt[3]{G^2}}{r^2} + 6.5\frac{G}{r^3}, \tag{2.136}$$

where Δp_f is the excess pressure on the front of the shock wave, in kg/cm^2; G the weight of the charge, in kg; and $\underline{r}$ the distance, in meters.

If it is recognized that the weight G can be expressed in terms of the radius of the equivalent spherical charge R_{03},** then Formula

(2.136) can be represented in the form

$$\Delta p_\Phi = 14{,}3\frac{1}{\bar r} + 907\frac{1}{\bar r^2} + 43\,600\frac{1}{\bar r^3}, \qquad (2.137)$$

where $\bar r = r/R_{03}$.

The time of action of the positive phase of the pressures is char-acterized by the relation

$$t_+ = 1{,}5\sqrt[6]{G}\sqrt{\bar r}\,10^{-3}\ \text{sec}, \qquad (2.138)$$

or, what is the same,

$$t_+ = 6{,}5 \cdot 10^{-3}\,R_{03}\sqrt{\bar r}. \qquad (2.139)$$

From Formulas (2.138)-(2.139) it follows that as the shock wave propagates the duration of the compression phase increases.

This fact can be readily explained by physical considerations. Indeed, the wave front moves in space with velocity $N > a_0$, whereas its "tail," in which the excess pressure is equal to zero (Fig. 47, region b-b'), moves with the velocity of sound a_0. Therefore as the wave moves away from the center of the explosion it stretches out in time. But the spatial extent of the compression zone remains in this case practically constant, since the increase in the time of action $\underline{t}$ occurs simultaneously with a drop in the displacement velocity N.*
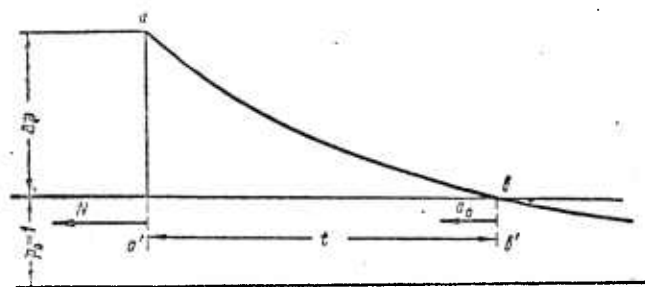
Fig. 47. Pressure pattern in an aerial shock wave.

Measurements of the impulse of the compression zone, also made by M.A. Sadovskiy, indicate that over a rather wide range of distances the following relation is true
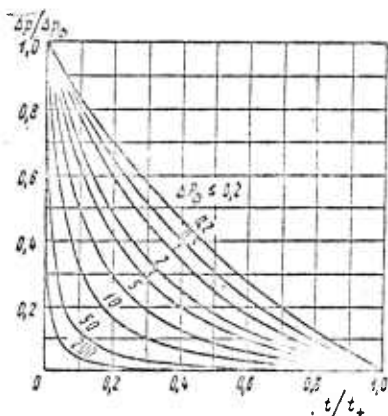
Fig. 48. Variation of
the pressure with time
for shock waves of
varying intensity
(at various distances
from the center of the
explosion).

$$I_+ = A \frac{G^{\frac{2}{3}}}{r}, \qquad (2.140)$$

where I_+ is the impulse of the compression
zone in kg-sec/m^2 and A is a constant coef-
ficient, which for TNT is approximately
equal to 34-36.

The impulse of the wave in the rarefac-
tion wave is determined from the empirical
formula

$$I_- = I_+ \left(1 - \frac{1}{2r}\right), \qquad (2.141)$$

from which it is seen that as the shock

wave propagates the value of the impulse of

the wave in the rarefaction phase approaches the value in the compres-

sion phase.

The character of the shock-wave pattern depends on the pressure

on the front. An illustrative idea on this question can be gained from

Fig. 48, which shows a series of curves for a wide range of Δp_f as a

function of the dimensionless time t/t_+. The approximation of these

data is given by the following formulas:*

$$\Delta p(t) = \Delta p_\Phi \left(1 - \frac{t}{t_+}\right) e^{-\frac{at}{t_+}} \qquad (2.142)$$

for 1 atm $< \Delta p_f <$ 3 atm, where

$$a = \frac{1}{2} + \Delta p_\Phi \left[1,1 - (0,13 + 0,20 \Delta p_\Phi) \frac{t}{t_+}\right], \qquad (2.143)$$

if

$$\Delta p_\Phi < 1 \text{ atm},$$

then

$$a = \frac{1}{2} + \Delta p_\Phi.$$

In the latter case we can also use, without great error, the lin-

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ear relation

$$\Delta p\,(t) = \Delta p_{\phi}\Big(1 - \frac{t}{t_+}\Big).\qquad(2.144)$$

Proceeding to the estimate of the pressure fields in the case of an explosion on the surface of the earth, it must be borne in mind that the very large difference between the densities of the air and of the ground allow us to regard the earth's surface as an absolutely rigid partition.

Thus, whereas it was previously assumed that the entire explosion energy is distributed in definite fashion in an unbounded space, in the presently considered case the energy will be distributed only in a half space. From this point of view, an explosion at the earth's surface is equivalent to an explosion of a charge of double the weight in the free atmosphere.

We then have for a surface explosion

$$\Delta p = 0{,}95\frac{\sqrt[3]{G}}{r} + 3{,}9\frac{\sqrt[3]{G^2}}{r^2} + 13{,}0\frac{G}{r^3};\qquad(2.145)$$

$$\Delta p = 17{,}9\frac{1}{r} + 1390\frac{1}{r^2} + 87\,200\frac{1}{r^3};\qquad(2.146)$$

$$t_+ = 1{,}7\cdot10^{-3}\sqrt[6]{G}\,\sqrt{r};\qquad(2.147)$$

$$t_+ = 7{,}0\cdot10^{-3}R_{03}\,\sqrt{r};\qquad(2.148)$$

$$I_+ = (52 \div 56)\frac{G^{2/3}}{r}.\qquad(2.149)$$

We note that some information regarding more complicated boundary problems, together with an approximate estimate of the values of the loads imposed on buildings and structures in the case of aerial explosions, will be given later on.

Example. Separate the main parameters of an aerial shock wave at a distance of 100 m from the center of a surface explosion of a charge of TNT weighing 200 kg.

Solution. Calculation of the excess pressure on the front of the wave is carried out in accordance with Formula (2.146):

$$\Delta p_\phi = 17.9\,\frac{1}{r} + 1390\,\frac{1}{r^2} + 87\,200\,\frac{1}{r^3}.$$

The radius of the equivalent spherical charge will be

$$R_{03} = 0.053\,\sqrt[3]{200} = 0.053 \cdot 7.368 = 0.39\ \text{м},$$

and therefore

$$\bar{r} = \frac{r}{R_{c3}} = \frac{100}{0.39} = 257;$$

$$\Delta p_\phi = 17.9\,\frac{1}{257} + 1390\,\frac{1}{257^2} + 87\,200 \cdot \frac{1}{257^3} =$$
$$= 0.0698 + 0.0211 + 0.00514 = 0.0958\ \text{kg/cm}^2.$$

The hydrodynamic parameters on the front of the shock wave will be calculated with the aid of the dynamic compatibility conditions (1.193)-(1.197):

$$N = 340\sqrt{1 + 0.83\Delta p_\phi} = 340\sqrt{1 + 0.83 \cdot 0.0958} = 351\ \text{m/sec};$$

$$v_\phi = \frac{235 \cdot \Delta p_\phi}{\sqrt{1 + 0.83\Delta p_\phi}} = \frac{235 \cdot 0.0958}{1.039} = 21.7\ \text{m/sec};$$

$$\rho_\phi = 0.125\,\frac{6 \cdot \Delta p_\phi + 7.2}{\Delta p_\phi + 7.2} = 0.125\,\frac{6 \cdot 0.0958 + 7.2}{0.0958 + 7.2} = 0.133\ \text{kg-sec}^2/\text{m}^4$$

$$T_\phi = 288\,\frac{(1 + \Delta p_\phi)(\Delta p_\phi + 7.2)}{6\Delta p_\phi + 7.2} = 288\,\frac{(1 + 0.0958)(0.0958 + 7.2)}{6 \cdot 0.0958 + 7.2} = 297°\ \text{K};$$

$$a_\phi = 20.1\sqrt{T_\phi} = 20.1\sqrt{297} = 345\ \text{m/sec}.$$

The variation of the pressure as a function of the time is determined from the formula

$$\Delta p(t) = \Delta p_\phi \left(1 - \frac{t}{t_+}\right),$$

with

$$t_+ = 7.3 \cdot 10^{-3} R_{03}\sqrt{\bar{r}} = 7.3 \cdot 10^{-3} \cdot 0.39\sqrt{257} = 0.045\ \text{sec}.$$

The value of the total impulse will be

$$I_+ \simeq 55\,\frac{G^{2/3}}{r};$$

$$I_- \simeq 55\,\frac{\sqrt[3]{200^2}}{100} = 18.8\ \text{kg-sec/m}^2.$$

§8. APPROXIMATE ESTIMATE OF THE PRESSURE FIELDS IN THE CASE OF AN UNDERWATER EXPLOSION IN AN UNBOUNDED LIQUID

It has been established as a result of many experiments that the pressure on the front of an underwater shock wave, in the absence of

any effect on the part of the bottom of the reservoir and of the free surface, is determined by the relation

$$p_m = A\left(\frac{G^{1/3}}{r}\right)^{1.13},\tag{2.150}$$

where A is a constant coefficient with a value of 533 for TNT.*

For dimensionless distances this relationship has the form

$$p_m = \frac{14\,700}{\bar{r}^{1.13}}\,\gamma_r.\tag{2.151}**$$

The variation of pressure with time is characterized in the middle range of distances ($60 < \bar{r} < 120$) by the relation

$$p(t, r) = p_m \frac{1}{\left[\alpha\left(\frac{ta_0}{R_{03}} - \frac{r}{R_{03}}\right) + 1\right]^2}\, \sigma_0\left(\frac{ta_0}{R_{03}} - \frac{r}{R_{03}}\right)\ \text{kg/cm}^2,\tag{2.152}$$

where α is a constant coefficient with value 0.27 for TNT and σ_0 is the unit discontinuity function of zero order, indicating the fact that the pressure increases abruptly on approaching the given point;

$$\sigma_0\left(\frac{ta_0}{R_{03}} - \frac{r}{R_{03}}\right) = \begin{cases} 0\ \text{for}\ \frac{ta_0}{R_{03}} < \frac{r}{R_{03}} \\ 1\ \text{for}\ \frac{ta_0}{R_{03}} \geq \frac{r}{R_{03}} \end{cases}$$

The value of the impulse of the positive pressure phase as a function of the time is

$$I(t, r) =$$
$$= p_m \frac{R_{03}}{a_0^n}\left\{1 - \frac{1}{\alpha\left(\frac{ta_0}{R_{03}} - \frac{r}{R_{03}}\right) + 1}\right\}\sigma_0\left(\frac{ta_0}{R_{03}} - \frac{r}{R_{03}}\right)\ \text{kg-sec/cm}^2,\tag{2.153}$$

The length of the underwater shock wave is approximately equal to 20 charge radii.

To estimate the variation of the pressure on the shock wave one uses, in addition, the exponential relationship

$$p = p_m e^{-\frac{1}{\theta}\left(t - \frac{r}{a_0}\right)}\sigma_0\left(t - \frac{r}{a_0}\right).\tag{2.154}$$

The quantity θ is called the time constant and is calculated with the aid of the relation

$$\theta = \left(\frac{G^{1/4}}{r}\right)^{-0,24} G^{1/4} \cdot 10^{-4} \text{ sec}, \qquad (2.155)$$

or

$$\theta = 1,4 \frac{R_{03}}{a_0} \overline{r}^{0,24} \text{ sec}. \qquad (2.156)$$

It is easy to note that when the distance increases the time constant θ increases. Thus, unlike (2.152), Formula (2.154) takes into account the fact that the pressure pattern stretches out in time as the shock wave moves away from the center of the explosion.

In accordance with (2.154), the pressure impulse will be

$$I = \int_0^t p\,dt = p_m \theta \left[1 - e^{-\frac{1}{\theta}\left(t - \frac{r}{a_0}\right)}\right] \sigma_0 \left(t - \frac{r}{a_0}\right). \qquad (2.157)$$

An account of the deformation of the pressure pattern can be made also when the pattern is approximated by a hyperbolic relationship. This is accomplished by using a relation of the form

$$p(t,r) = \frac{1}{\left[1 + 0,6 \frac{1}{\theta}\left(t - \frac{r}{a_0}\right)\right]^2} \sigma_0 \left(t - \frac{r}{a_0}\right) \qquad (2.158)$$

and accordingly

$$I(t,r) = p_m \frac{t - \frac{r}{a_0}}{1 + 0,6 \frac{1}{\theta}\left(t - \frac{r}{a_0}\right)} \sigma_0 \left(t - \frac{r}{a_0}\right). \qquad (2.159)$$

The change in pressure behind the front is determined most accurately by Formula (2.158).

In many problems, however, it is more convenient to use the exponential relation (2.154), and this is why the latter has gained the widest application.

We have already mentioned earlier that the energy flux density is given by the expression

$$E = \int_0^t \rho \vec{v} \Delta \left(\frac{u}{A} + \frac{1}{2} v^2 + \frac{p}{\rho}\right) dt. \qquad (1.179)$$

When the pressures on the front of the underwater shock wave are on the order of thousands of atmospheres, the internal and kinetic energies can be neglected compared with the term p/ρ. Inasmuch as the velocity of particle motion can in this case be calculated in the acoustic approximation

$$v = \frac{1}{\rho_0 a_0} (p - p_0) + \frac{1}{\rho_0 r} \int_0^t (p - p_0)\, dt,\qquad (2.134)$$

the expression for the energy flux density is obtained in the form

$$E = \frac{1}{\rho_0 a_0} \int_0^t (p - p_0)^2\, dt + \frac{1}{\rho_0 r} \int_0^t (p - p_0)\left[\int_0^{t'} (p - p_0)\, dt'\right] dt.\qquad (2.160)$$

Calculations show that the second term rapidly becomes negligibly small compared with the first as the distance $\underline{r}$ from the center of the explosion increases. For the indicated range of pressures (p < 1000 atm) we then obtain with sufficient degree of accuracy

$$E = \frac{1}{\rho_0 a_0} \int_0^t (p - p_0)^2 v\, dt.\qquad (2.161)$$

The formulas presented cover fully the question of estimating the pressure fields in the case of an explosion produced by a spherical charge in an unbounded liquid.

A cylindrical wave attenuates much more slowly with distance than a spherical one. The exponent in this case is approximately equal to 0.56 and

$$p_{\text{цил}} = B \frac{1}{r^{0.56}}.$$

The amplitude of a plane wave decreases even more slowly with the distance.

Comparison of the character of the variation of the pressure on the front of a plane, cylindrical, and spherical wave is shown in Fig. 49. The difference between them, as can be readily seen, is quite ap-
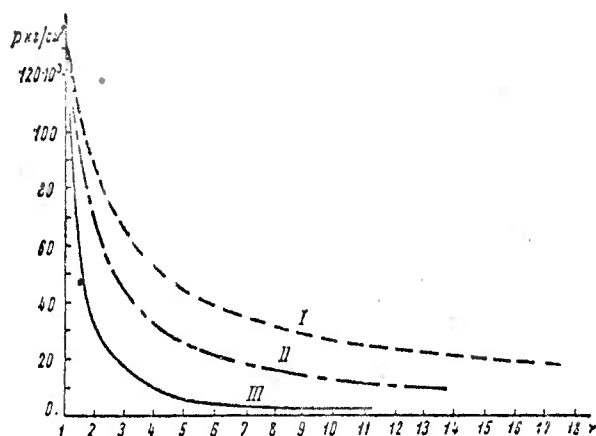
Fig. 49. Comparison of the character of the variation of the pressure on the front of a plane (I), cylindrical (II), and spherical (III) shock wave.

preciable. The rather slow variation of the amplitude of the plane wave with distance is the reason why sound propagates over superhigh distances in the ocean* and is extensively used in the operating practice of hydroacoustic stations.

In conclusion we emphasize that the data presented pertain only to the first pulsation of the gas bubble. The second and succeeding pulsations also give rise to shock waves, but their intensities are much lower. For a general estimate one can note that the impulses of the pressures produced in the second pulsation are five to six times smaller than in the first pulsation; in the third pulsation they are, respectively, three times smaller than in the second.

Example. Plot the pressure pattern of an underwater shock wave at a point located at a distance 100 m from the center of the explosion of a TNT charge weighing 1 ton. Carry out the calculation both with the hyperbolic and with the exponential dependences.

Solution. We find the radius of the equivalent spherical charge and the relative distance

$$R_{03} = 0.053 \sqrt[3]{1000} = 0.53 \text{ m};$$

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$$\bar{r} = \frac{r}{R_{03}} = \frac{95}{0,53} = 179.$$

We calculate the value of p_m

$$p_m = \frac{14\,700}{\bar{r}^{1,13}} = \frac{14\,700}{179^{1,13}} = 42,5 \ \text{kg/cm}^2.$$

Calculation of the variation of the pressure of the underwater shock wave is carried out by Formula (2.152)

$$p\,(t,\,r) = p_m \frac{1}{\left[\,a\left(\frac{ta_0}{R_{03}} - \frac{r}{R_{03}}\right) + 1\,\right]^2}\,\sigma_0\left(\frac{ta_0}{R_{03}} - \frac{r}{R_{03}}\right)\ \text{kg/cm}^2.$$

The calculation is summarized in Table 9.

We determine the damping time constant θ by Formula (2.156)

$$\theta = 1,4\frac{R_{03}}{a_0}\cdot\bar{r}^{0,24} = \frac{1,4\cdot0,53\cdot3,46}{1500} = 0,00171\,\text{sec}.$$

We calculate the maximum pressure using the exponential function (2.154)

$$p = p_m e^{-\frac{1}{\theta}\left(t - \frac{r}{a_o}\right)}\,\sigma\left(t - \frac{r}{a_0}\right).$$

The calculation data are listed in Table 10.

We calculate the values of the maximum pressure using the hyperbolic dependence (2.158)

$$p\,(t,\,r) = \frac{p_m}{\left[1 + 0,6\frac{1}{\theta}\left(t - \frac{r}{a_0}\right)\right]^2}\,\sigma_0\left(t - \frac{r}{a_0}\right).$$

This calculation is summarized in Table 11.

Figure 50 shows a comparison of the calculation results obtained from all three relations employed. Since the relative distance $\bar{r} = 179$ is beyond the range of possible application of Formula (2.152), the latter yields the largest error. The exponential curve is close to hyperbolic for time intervals $t < \theta$.

TABLE 9

t, sec	ta_0	$(2)/R_{03}$	$(3) - 179$	$(4)\cdot 0{,}27$	$(5) + 1$	$(6)^2$	$\dfrac{1}{(7)}$	$p = p_m(8) = 42{,}5(8)$
1	2	3	4	5	6	7	8	9
0,0633	95,0	179	0	0	1,00	1,00	1,000	42,50
0,0640	96,0	181	2,0	0,540	1,54	2,37	0,422	17,90
0,0645	96,8	182	3,5	0,945	1,94	3,78	0,265	11,25
0,0650	97,5	184	5,0	1,350	2,35	5,52	0,181	7,68
0,0655	98,3	185	6,0	1,620	2,62	6,86	0,146	6,22
0,0660	99,0	187	8,0	2,160	3,16	9,98	0,100	4,25
0,0665	99,7	188	9,0	2,430	3,43	11,76	0,0852	3,62
0,0670	100,0	189	10,0	2,700	3,70	13,69	0,0732	3,11
0,0700	105,0	198	19,0	5,130	6,13	37,58	0,0267	1,13
0,0720	108,0	204	25,0	6,750	7,75	60,06	0,0167	0,71

TABLE 10

t, sec	$t - \dfrac{r}{a_0}$	$\dfrac{1}{\theta}\,(2)$	$e^{-(3)}$	$p = p_m(4) = 42{,}5(4)$
1	2	3	4	5
0,0633	0	0	1,0000	42,5
0,0640	0,0007	0,410	0,6636	28,1
0,0645	0,0012	0,703	0,4951	21,0
0,0650	0,0017	0,995	0,3697	15,6
0,0655	0,0022	1,290	0,2752	11,7
0,0660	0,0027	1,580	0,2059	8,73
0,0665	0,0032	1,870	0,1541	6,55
0,0670	0,0037	2,160	0,1153	4,90
0,0700	0,0067	3,920	0,0198	0,84
0,0720	0,0087	5,100	0,0061	0,26

TABLE 11

t, sec	$\dfrac{1}{\theta}\left(t - \dfrac{r}{a_0}\right)$	$0{,}6\,(2)$	$1 + (3)$	$(4)^2$	$p = \dfrac{p_m}{(5)} = \dfrac{42{,}5}{(5)}$
1	2	3	4	5	6
0,0633	0	0	1,000	1,000	42,5
0,0640	0,410	0,246	1,246	1,550	27,4
0,0645	0,703	0,422	1,422	2,016	21,1
0,0650	0,995	0,597	1,597	2,550	16,6
0,0655	1,290	0,774	1,774	3,140	13,5
0,0660	1,580	0,948	1,948	3,790	11,2
0,0665	1,870	1,120	2,120	4,500	9,4
0,0670	2,160	1,295	2,295	5,250	8,1
0,0700	3,920	2,350	3,350	11,200	3,8
0,0720	5,100	3,060	4,060	16,480	2,6
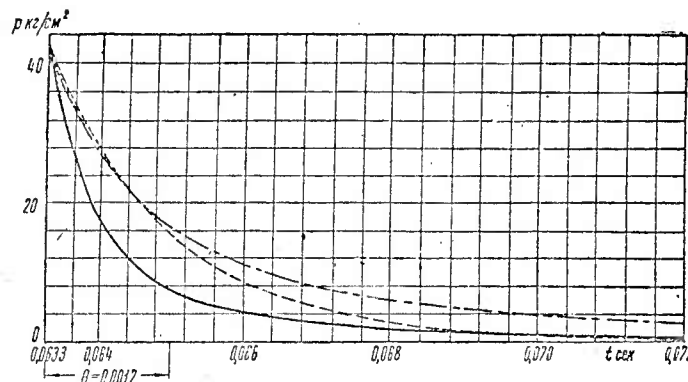
Fig. 50. Comparison of the values of
the maximum pressure determined from
Formula (2.152) (solid line), from the
exponential formula (2.154) (dashed
line), and from the hyperbolic depend-
ence (2.158) (dash-dot). Time is
reckoned from the instant of explosion.

§9. PROPAGATION OF LOAD WAVE IN AN UNDERGROUND EXPLOSION IN AN UNBOUNDED MEDIUM

The initial period of the development of the processes that accom-
pany an underground explosion essentially does not differ much from
the case of an underwater explosion. At the instant when the detona-
tion wave emerges on the surface of the charge, three discontinuity
surfaces are produced: a shock wave, a stationary strong-discontinuity
surface, and a weak-discontinuity surface (characteristic).

The pressure on the front of the shock wave is on the order of
hundreds of thousands of atmospheres. At such high loads, the ground
behaves like a gas, the only difference being that after the pressure
is removed the gas returns to its initial state, whereas the ground
either disintegrates or experiences large residual deformations.

The disintegration and packing together of the ground consumes
irreversibly considerable part of the explosion energy. Therefore the
pressure on the wave front decreases rapidly. At a distance of two or
three large radii, the maximum pressure amounts to only several thou-
sand atmospheres.

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Also coincident with this time is the beginning of a qualitative change in the process of propagation of the wave disturbances. The point is that at stresses of this order, unlike in an ideal liquid, the propagation velocity of a compression wave of large amplitude in ground is lower than the velocity of sound. Consequently, the wave pattern becomes deformed in the compression phase. The wave front disintegrates and is replaced by a gradual increase in pressure. The region of the greatest stresses lags the start of the arrival of the load wave.

As before, however, in this range of distances one still uses the term "wave front," taking this to mean the envelope surface of the initial disturbances of the compression wave. The rate of propagation of such a front is obviously equal to the rate of propagation of the longitudinal waves in the ground:

$$c = \sqrt{\frac{\lambda + 2\mu}{\rho}}, \qquad (2.162)$$

where λ and μ are the Lame elastic constant and ρ is the density.

As the compression wave moves away from the center of the explosion, the maximum stresses in it decrease, and the disintegrations and plastic deformations of the ground are replaced by elastoplastic deformations and then simply by elastic deformations.

In accordance with the qualitative picture developed above, it is customary to distinguish between three zones of an underground explosion: the near zone, the middle zone, and the far zone.*

Their dimensions are determined essentially by the weight of the charge and by the type of ground.

In the near zone one separates two other regions, the compression region and the disintegration region. In the compression region there exists a shock wave with discontinuous front. Owing to the tremendous

loads the ground crumbles and becomes packed together. The boundary of this zone, in the case of explosion of TNT charges, is determined from the empirical formula

$$\overline{R}_s = 19k_s, \qquad (2.163)$$

where $\overline{R}_s$ is the relative radius of the compression region, $\overline{R}_s = R_s/R_{03}$, and k_s is an empirical coefficient which depends on the type of ground.

The region of disintegration is characterized by the disintegration of the continuity of the ground and by the appearance of cracks.

For the disintegration region we have

$$\overline{R}_r = 19.0k_r, \qquad (2.164)$$

where $\overline{R}_r = R_r/R_{03}$ is the relative radius of the disintegration sphere and k_r is an empirical coefficient.

The values of the coefficients k_s and k_r are listed in Table 12.

The laws governing the propagation of the compression wave in the near zone have not yet been studied sufficiently to this day. Consequently, one cannot even indicate general formulas to characterize the stress and velocity fields for the ground particles in this zone.

The middle zone of the underground explosion, where elastoplastic ground deformations occur, extends approximately to distances of about $100\ R_{03}$.

For soft grounds of the loess and loam type, etc., the normal stresses $\sigma_{r_{max}}$ in the middle zone can be calculated from the formula*

$$\sigma_{r_{max}} = \frac{425\,Fk}{R^3}\ [\text{kg/cm}^2], \qquad (2.165)$$

where the coefficient F for large depths of placement of the charge can be assumed equal to unity. The value of the coefficient k, together with some other characteristics of grounds, is listed in Table 13.

In soft water-saturated grounds one usually assumes the normal

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TABLE 12

1 Наименование грунта	k_c	k_p
2 Грунт рыхлый, свеженасыщенный	0,60	0,85
3 Песок плотный	0,50	0,63
4 Глина	0,50	0,60
5 Скальные породы	0,20	0,50

1) Name of ground; 2) loose, freshly saturated ground; 3) dense sand; 4) clay; 5) rocks.

TABLE 13

1 Вид грунта	$k_{min},$ $кг/см^2$ 7	$k_{max},$ $кг/см^2$	$k_{среди},$ $кг/см^2$	$a_{min},$ $м/сек$ 8	$a_{max},$ $м/сек$	$\rho,$ $кгсек^2/м^2$ 9
2 Лёсс	30	120	60	140	600	160—180
3 Илистая глина	90	630	360	130	900	170—190
4 Суглинок	90	180	140	150	300	170—190
5 Ненасыщенная глина	700	1 400	1 050	800	1 700	182
6 Насыщенная глина	3 500	10 600	7 000	1 700	2 500	200—230

1) Type of ground; 2) loess; 3) silty clay; 4) loam; 5) unsaturated clay; 6) saturated clay; 7) kg/cm^2; 8) m/sec; 9) kg-sec^2/m^2.

and tangential stresses to be close in value, i.e., $\sigma_{r_{max}}/\sigma_{\varphi_{max}} \simeq 1$.

Consequently, Formula (2.165) determines approximately the entire field of stresses in soft grounds in the case of an underground explosion.

For rocks such as granite, marble, limestone, etc., the largest normal stress can be calculated from the empirical formula obtained by Khanukayev

$$\sigma_{r_{max}} = \frac{10^4 \cdot F}{\rho a}\left[\frac{95,1}{\bar{R}} - \frac{70,1 \cdot 10^2}{\bar{R}^2} + \frac{26,6 \cdot 10^4}{\bar{R}^3}\right] \text{kg/cm}^2, \qquad (2.166)$$

where F is a coefficient that depends on the depth of the explosion, which for large depths of placement of the charge is equal to unity, ρ is the density of the rock in kg-sec^2/m^4, and a is the velocity of sound in the rock, in m/sec.

The values of the velocity of propagation of longitudinal and transverse waves in rocks are listed in Table 14.

TABLE 14

1 Вид породы	2 Гранит	3 Мрамор	4 Диабаз
5 Скорость продольных волн c, *м/сек*	5200—5500	4400—4800	5000
6 Скорость поперечных волн b, *м/сек*	3100—3200	3500—2800	3200
7 Плотность породы ρ, *кг сек²/м⁴*	265	265	273

1) Type of rock; 2) granite; 3) marble; 4) diabase; 5) velocity of longitudinal waves c, m/sec; 6) velocity of transverse waves b, m/sec; 7) density ρ of rock, kg-sec^2/m^4.

The ratio of the normal and tangential stresses at considerable distances from the center of the explosion (when $1/\overline{R}^2 \ll 1/\overline{R}$) is approximately

$$\frac{\sigma_r}{\sigma_\varphi} \approx \frac{1}{1-2\frac{b^2}{a^2}}. \qquad (2.167)$$

The maximum values of the velocities of motion of the ground can be approximately determined from the relation

$$\sigma_{r_{max}} = v_{r_{max}}\,\rho a \qquad (2.168)$$

so that in accordance with (2.165) and (2.166), we have:

for soft grounds

$$v_{r_{max}} = \frac{425\,Fk}{\rho a\,\overline{R}^3}\,10^6 \text{ cm/sec;} \qquad (2.169)$$

for rocks

$$v = \frac{10^{12}F}{\rho^2 a^2}\left[\frac{95,1}{\overline{R}} - \frac{70,1\cdot 10^2}{\overline{R}^2} + \frac{26,6}{\overline{R}^3}\,10^4\right] \text{ m/sec.} \qquad (2.170)$$

In individual problems it may be of interest to determine the impulse of the load wave. According to American data, for soft grounds and an underground explosion of a TNT charge we have

$$\frac{I}{R_{03}} = \frac{3770\,Fk'_{cp}}{\overline{R}^{7/4}} \text{ kg-sec/m}^3, \qquad (2.171)$$

TABLE 15

1 Вид грунта	6 $k'_{ср}$, кг сек/м²
2 Лёсс	1 130
3 Суглинок	3 360
4 Нанесая глина	3 620
5 Глина	4 660

1) Type of ground; 2) loess; 3) loam; 4) silty clay; 5) clay; 6) k'_{sr}, kg-sec/m².

where k'_{sr} is an empirical coefficient, the values of which are listed in Table 15.

In the case of an explosion in solid rock

$$\frac{I}{R_0} = F\left[\frac{42,75}{R} - \frac{25,5\cdot10^2}{R^2} + \frac{6,3\cdot10^4}{R^3}\right]10^4 \text{ kg-sec/m}^3. \quad (2.172)$$

Knowing the value of the normal stress and the impulse in the compression phase, we can approximately estimate the time of action of the load. Assuming the pressure pattern to be triangular, we obtain

$$I_r \approx \frac{1}{2}\sigma_{r_{max}}T,$$

hence

$$T \approx \frac{2I_r}{\sigma_{r_{max}}} \text{ sec.} \qquad (2.173)$$

The buildup time of the load, for solid grounds, is approximately (1/5-1/3)T. For soft grounds this time fluctuates in the range (1/4-1/2)T.

The far zone of an underground explosion is arbitrarily confined to the range of distances from 100 to 1000 R_{03}. The stresses and particle velocities in this zone are characterized approximately by the same relationships as in the middle zone [Formulas (2.165)-(2.170)].

The relations presented give the general pattern of propagation of seismic-explosion waves when the charge is placed at a sufficiently great depth. Explosions of this type, which display no visible action on the free surface, are frequently called camouflets.

The presence of a free surface greatly influences the stress and velocity fields in the case of an underground explosion. Some data on this question will be given at the end of the next chapter.

Example. Calculate the values of the stresses and velocities of motion of the ground upon explosion of a TNT charge weighing 100 tons in a rock of the type of granite at a distance of 250 m from the center of the explosion. Determine the time of action of the load.

Solution. We find the radius of the equivalent spherical charge R_{03} and the relative distance of the point from the center of the explosion:

$$R_{03} = 0.053 \sqrt[3]{100 \cdot 10^4} = 0.053 \cdot 10 \cdot 4.61 = 2.46 \text{ m},$$

$$\overline{R} = \frac{R}{R_{03}} = \frac{250}{2.46} = 102.$$

The investigated point is in the middle zone of the explosion.

We determine σ_{max} from Formula (2.166). The values of ρ and a are obtained from Table 14. In our case $F \approx 1$, $\rho = 265$ kg-sec^2/m^4, and $a = 5350$ m/sec:

$$\sigma_{r_{max}} = \frac{10^4 \cdot F}{\rho a} \left[\frac{95.1}{R} - \frac{70.1 \cdot 10^2}{R^2} + \frac{26.6 \cdot 10^4}{R^3} \right] =$$

$$= \frac{10^4 \cdot 1}{265 \cdot 5350} \left[\frac{95.1}{102} - \frac{70.1 \cdot 10^2}{102^2} + \frac{26.6 \cdot 10^4}{102^3} \right] =$$

$$= \frac{10^4}{10^5 \cdot 14.2} [0.932 - 0.674 + 0.251] = 70.4 \cdot 0.509 = 35.8 \text{ kg/cm}^2.$$

We find $v_{r_{max}}$ from Formula (2.170)

$$v_{r_{max}} = \frac{10^{12} \cdot F}{\rho^2 a^2} \left[\frac{95.1}{R} - \frac{70.1 \cdot 10^2}{R^2} + \frac{26.6 \cdot 10^4}{R^3} \right] = 0.252 \text{ m/sec.}$$

We determine the time of action of the load T from Formula (2.173). For this purpose we calculate the impulse of the wave in the compression phase using Formula (2.172):

$$\frac{I}{R_{03}} = F \left[\frac{42.8}{R} - \frac{25.5 \cdot 10^2}{R^2} + \frac{6.3 \cdot 10^4}{R^3} \right] 10^4 = 2.34 \cdot 10^3 \text{ kg-sec/m}^3,$$

and then

$$T \simeq \frac{2I_r}{\sigma_{r_{max}}} = \frac{2 \cdot 2.34 \cdot 10^3 \cdot 2.46 \cdot 10^{-4}}{35.8} = 0.0321 \text{ sec.}$$

113 Equation (2.16) is the consequence of Formula (1.58) in §4 of Chapter 1.

113 Analogous relations in the variables $\theta = (p/\rho^k)^{1/k}$, v, and N were obtained by I.P. Ginzburg (see N.Ye. Kochin, I.A. Kibel', N.V. Roze, Theoretical Hydromechanics, Vol. II, OGIZ, 1948).

114 L.D. Landau and K.P. Stanyukovich, DAN, Vol. 47, 1945, Nos. 3 and 4.

116 F.S. Baum, K.P. Stanyukovich, B.I. Shekhter, Physics of Explosions, Fizmatgiz, 1959.

126 See the examples of the preceding section.

127 It can be shown that the conditions of dynamic compatibility (2.42)-(2.44) are of accuracy sufficient for practice if $a_0/N < 0.1$.

134 See Eq. (2.56a).

135 When $t^* = 1/N^*$ and $r_0 = 1$.

139 See §8 of the present chapter.

139 See §5 of Chapter 1, Eqs. (1.97) and (1.98).

145 L.D. Landau, Shock Waves at Long Distances from the Point of Their Occurrence. Applied Mathematics and Mechanics, Vol. IX, No. 4, 1945.

154 In accordance with the principle of energy similarity, for an approximate estimate of the pressure field due to an explosion of another explosive, we can use Formula (2.136), the only difference being that the quantity G must be replaced by GQ_i/Q_t, where Q_i is the specific energy of the given explosive in kcal/kg, Q_t is the specific energy of TNT, with $Q_t = 1000$ kcal/kg.

154 $R_{03} = 0.053 \sqrt[3]{G}$ (2.21).

155 M.A. Sadovskiy notes that as the distance is increased a certain increase is observed in the wavelength, characterized by the approximate relation $\lambda = \lambda_0 + a \ln r$.

156 F.S. Baum, B.I. Shekhter, K.P. Stanyukovich, Physics of Explosions, GIFML, 1959.

[Footnotes (Continued)]

159   According to the principle of energy similarity, the value of the coefficient A_i for other explosives can be calculated with the aid of the approximate relationship $A_i = \gamma_i A$

$$\gamma_i = \left(\frac{Q_i}{Q_\tau}\right)^{\frac{1.13}{3}} = \left(\frac{Q_i}{Q_\tau}\right)^{0.370}.$$

where Q_i is the specific energy of the given substance in kcal/kg, and Q_t is the specific energy of TNT, $Q_t = 1000$ kcal/kg.

159   For TNT, obviously, $\gamma = 1$.

162   "Propagation of Sound in the Ocean," Collection of Articles, IIL, 1951.

166   Such a classification is quite arbitrary in character. In many publications, a different terminology is used. For example, V.A. Assonov distinguishes between a compression sphere, a crumbling sphere, and a quake sphere.

167   The Effects of Atomic Weapons, New York-Toronto-London, 1950.

[List of Transliterated Symbols]

116   ф = f = front = front

119   н = n = naturnyy = natural

119   м = m = model'nyy = model

161   цил = tsil = tsilindricheskaya volna = cylindrical wave

167   с = s = szhatiye = compression

167   р = r = razrusheniye = disintegration

168   средн = sredn = sredniy = average

169   ср = sr = sredniy = average

Chapter 3

SIMPLEST BOUNDARY PROBLEMS OF EXPLOSION THEORY

§1. REFLECTION OF AN AERIAL SHOCK WAVE FROM THE SURFACE OF THE EARTH

As was already noted, under suitable angles of incidence, the reflected wave propagating in a perturbed medium catches up with the direct wave, merges with the latter, forming a frontal wave so that irregular reflection sets in.

In spite of the principal clarity in the formulation of the corresponding boundary problem, the presently obtained theoretical solutions, for both the region of regular and particularly the region of irregular reflection, are quite cumbersome in form and in many cases they diverge substantially from the experimental data. The most complete experimental research in this field, which we shall present below, has been made by K. Ye. Gubkin and A. I. Korotkov.

In accordance with the Izmaylov formula, the pressure in the reflected wave at normal incidence ($\alpha = 0$) is equal to

$$p_{\text{отр}} - p_0 = 2(p_\phi - p_0) + \frac{\frac{k+1}{k-1}(p_\phi - p_0)^2}{(p_\phi - p_0) + \frac{2k}{k-1}p_0}, \qquad (1.220)$$

or, if we assume that $p_0 = 1$ atm and $k = 1.4$, then

$$\Delta p_{\text{отр}} = 2\Delta p_\phi + \frac{6\Delta p_\phi^2}{\Delta p_\phi + 7}. \qquad (1.221)$$

It has been found that in the region of irregular reflection, for amplitudes on the front not exceeding 3.0 kg/cm^2, the pressure in the reflected wave is practically independent of the angle of incidence and can be determined from Formula (1.221). For the region of irregular

reflection ($\alpha_{pred} \leq \alpha \leq 90^{\circ}$) the experimental data are approximated by the approximate relationship

$$\Delta p_{\alpha} = \Delta p'_{\phi} (1 + \cos \alpha),$$ (3.1)

where $\Delta p'_f$ is the pressure on the front of the shock wave of an explosion on land, calculated by the formula of M.A. Sadovskiy (2.145).

The experimentally obtained boundary of the reflection regions is shown in Fig. 51.

The laws governing the variation of the momentum in the compression phase with varying angle of incidence have a somewhat different character. As was shown by M.A. Sadovskiy, in the range $0^{\circ} < \alpha < 45^{\circ}$ we have

$$I = I_0 (1 + \cos \alpha)$$ (3.2)

and in the range $45^{\circ} < \alpha < 90^{\circ}$

$$I = I_0 (1 + \cos^2 \alpha),$$ (3.3)

where I_0 is the impulse for a land explosion, calculated by Formula (2.149).

The height of the explosion influences the pressure at the earth's surface in two ways: it increases the distance between the center of the explosion and the point of observation, and it changes the angle of incidence of the wave and accordingly the regions of regular and irregular reflection. It follows therefore that for a given wave intensity Δp_z it is possible to indicate a value of the height H such that the distance from the epicenter to the point with pressure Δp_z turns out to be maximum. This is called the optimum height.

If the charge is located above or below the optimum height H_{opt}, the dimensions of the radii of the zones of specified action (L_{p_z}) decrease, and when the height H is reduced compared with H_{opt} this influences the value of L_{p_z} much less than an increase in H compared with H_{opt}.
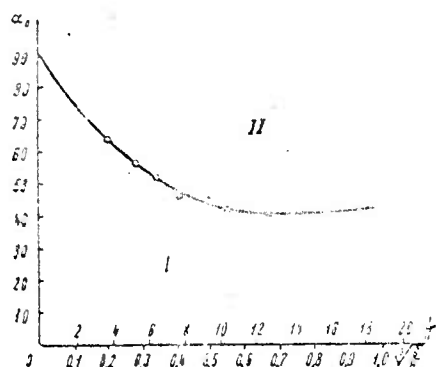
Fig. 51. Regions of regular (I) and irregular (II) reflection as obtained from the experimental data.

The empirical formula relating the optimum height with the weight of the charge and with the specified pressure on the wave front has the form

$$H_{\text{опт}} \simeq 3.2 \sqrt[3]{\frac{G}{\Delta p_3}}, \qquad (3.4)$$

where G is the weight of the charge in kg, Δp_z the specified value of the pressure on the front, kg/cm^2, and H_{opt} is the optimum height, in meters.

The horizontal distances L_{p_z} at which the specified value of the pressure p_z is observed in the case of an explosion at the optimum height are expressed by the approximate equation

$$L_{p_3} = 1.3 \frac{H_{\text{опт}}}{\Delta p_3^{0.4}}. \qquad (3.5)$$

M.A. Sadovskiy was the first to call attention to the fact that in the region of regular reflection, at equal true distances from the charge to the point of observation $r = (L^2+H^2)^{1/2}$, the values of the pressure are equal and do not depend on the angle of incidence α (i.e., the equation $\Delta p_{\substack{H=a \\ L=b}} = \Delta p_{\substack{H=b \\ L=a}}$ holds true).

Consequently, in the region of regular reflection the connection between Δp_{otr} and L and H is described with sufficient accuracy by Sadovskiy's three-term formula for an explosion in an unlimited atmosphere

$$\Delta p_{\phi} = 0.76 \frac{\sqrt[3]{G}}{\sqrt{L^2+H^2}} + 2.47 \frac{\sqrt[3]{G^2}}{L^2+H^2} + 6.5 \frac{G}{\sqrt{(L^2+H^2)^3}} \qquad (3.6)$$

and by the Izmaylov formula (1.221).

§2. EFFECT OF THE FREE SURFACE OF THE LIQUID ON THE PRESSURE FIELDS IN AN UNDERWATER EXPLOSION. ACOUSTIC APPROXIMATION FORMULAS

In underwater explosions at relatively shallow depths, the parameters of the hydrodynamic field are greatly influenced by the free surface of the water, on which the pressure is equal to atmospheric pressure. As was already noted, when a wave is incident on the free surface, a rarefaction wave is produced.

The problem of estimating the hydrodynamic fields with allowance for the influence of the free surface can be solved in different ways.

We shall develop here one of the solutions, with an aim at simultaneously illustrating, by means of a very simple example, the so-called method of incomplete separation of variables, which is widely used in the analysis of various boundary-value problems in the theory of the wave potential.

Assume that on some sphere of radius R = R* there is specified a pressure variation as a function of the time:

$$p\Big|_{R=R^\cdot} = \rho_0 \frac{\partial \varphi}{\partial t}\Big|_{R=R^\cdot} = p^* e^{-\frac{t}{\tau}}, \qquad (3.7)$$

and that the excess pressure on the free surface of the liquid is equal to zero

$$\rho_0 \frac{\partial \varphi}{\partial t} = 0\Big|_{z=0}. \qquad (3.8)$$

At the instant of time t = 0 the liquid is at rest

$$\varphi\big|_{t=0} = 0; \qquad (3.9)$$

$$\frac{\partial \varphi}{\partial t}\Big|_{t=0} = 0. \qquad (3.10)$$

The unsteady motion of the liquid at t > 0 is characterized by the wave equation

$$\Delta\varphi = \frac{1}{a_0^2}\frac{\partial^2 \varphi}{\partial t^2}, \qquad (3.11)$$

integration of which should be carried out under the above-formulated initial and boundary conditions.

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Let us consider first the construction of the solution for the direct wave.

Equation (3.11) for motions with spherical symmetry can be rewritten in the form

$$\frac{\partial^2 \varphi}{\partial R^2} + \frac{2}{R} \frac{\partial \varphi}{\partial R} = \frac{1}{a_0^2} \frac{\partial^2 \varphi}{\partial t^2}. \tag{3.12}$$

We assume that

$$\varphi_1 = R\varphi, \tag{3.13}$$

and then

$$\frac{\partial^2 \varphi_1}{\partial R^2} = \frac{1}{a_0^2} \frac{\partial^2 \varphi_1}{\partial t^2}. \tag{3.14}$$

We apply to Eq. (3.14) the one-sided Laplace transform. To this end we multiply both halves of the equation by e^{-st} and integrate from zero to infinity. At first we choose for the parameter $\underline{s}$ some complex number with positive real parts ($s = \sigma + i\tau$; $\sigma > 0$).

We have

$$\frac{\partial^2}{\partial R^2} \int_0^\infty \varphi_1 e^{-st} dt = \frac{1}{a_0^2} \int_0^\infty \frac{\partial^2 \varphi_1}{\partial t^2} e^{-st} dt. \tag{3.15}$$

We integrate the right half of (3.15) by parts

$$\int_0^\infty \frac{\partial^2 \varphi_1}{\partial t^2} e^{-st} dt = e^{-st} \frac{\partial \varphi_1}{\partial t} \Big|_0^\infty + s \int_0^\infty \frac{\partial \varphi_1}{\partial t} e^{-st} dt.$$

The first term of the right half vanishes because of the initial condition (3.10) and because of the exponential character of the first factor.

We continue the integration by parts:

$$s \int_0^\infty \frac{\partial \varphi_1}{\partial t} e^{-st} dt = s \left(e^{-st} \varphi_1 \right) \Big|_0^\infty + s^2 \int_0^\infty \varphi_1 e^{-st} dt. \tag{3.16}$$

Again we conclude that the first term vanishes.

We put $\qquad \qquad \Phi(s, R) = \int_0^\infty \varphi_1 e^{-st} dt. \tag{3.17}$

Then on the basis of (3.14)-(3.17) we obtain

$$\frac{\partial^2 \Phi}{\partial R^2} = \frac{1}{a_0^2} s^2 \Phi,$$

or

$$\frac{\partial^2 \Phi}{\partial R^2} - \frac{1}{a_0^2} s^2 \Phi = 0. \tag{3.18}$$

Thus, in place of the wave equation (3.14) we obtain with the aid of the Laplace transform an ordinary second-order differential equation. Its general integral is

$$\Phi(R,s) = c_1(s) e^{-\frac{1}{a_0} sR} + c_2(s) e^{\frac{1}{a_0} sR}. \tag{3.19}$$

It is obvious from physical considerations that as $R \to \infty$ the perturbations determined by the function $\Phi(R, s)$ should be equal to zero. Consequently, $c_2(s) \equiv 0$, and

$$\Phi(R,s) = c_1(s) e^{-\frac{1}{a_0} sR}. \tag{3.20}$$

We obtain the function $c_1(s)$ from the boundary condition on the spherical surface:

$$\rho_0 \frac{\partial \varphi}{\partial t}\bigg|_{R=R^*} = p(t)\big|_{R=R^*} = p^* e^{-\frac{t}{\theta}}. \tag{3.7}$$

To this end we again use the one-sided Laplace transform. We obtain

$$\rho_0 \int_0^\infty \frac{\partial \varphi}{\partial t} e^{-st} dt = \int_0^\infty p(t) e^{-st} dt.$$

We denote

$$\int_0^\infty p(t) e^{-st} dt = p^*(s). \tag{3.21}$$

We integrate the left half of (3.21) by parts

$$\rho_0 \int_0^\infty \frac{\partial \varphi}{\partial t} e^{-st} dt = \rho_0 e^{-st} \varphi\bigg|_0^\infty + \rho_0 s \int_0^\infty \varphi e^{-st} dt = \rho_0 s \int_0^\infty \frac{\varphi_1}{R} e^{-st} dt,$$

but $\Phi(s, R_0) = \int_0^\infty \varphi_1 e^{-st} dt$. Therefore

$$\rho_0 \int_0^\infty \frac{\partial \varphi}{\partial t} e^{-st}\, dt = \frac{\rho_0 s}{R} \Phi(s, R_0) = p^*(s)$$

or

$$\Phi(s, R^*) = \frac{p^*(s)}{\rho_0 s} R^*. \tag{3.22}$$

Comparing (3.20) with (3.22) we obtain

$$c_1(s) = \frac{p^*(s)}{\rho_0 s} R^* e^{\frac{1}{a_0} s R^*}.$$

Consequently,

$$\Phi(R, s) = R^* \frac{p^*(s)}{\rho_0 s} e^{-\frac{1}{a_0} s (R - R^*)}. \tag{3.23}$$

After determining the function $\Phi(R,\ s)$, we obtain the function φ_1. This is done by solving the integral equation

$$\Phi(R, s) = \int_0^\infty \varphi_1 e^{-st}\, dt. \tag{3.24}$$

Such a solution is known as the Mellin integral and is written in the form

$$\varphi_1(R, t) = \frac{1}{2\pi i} \int_l \Phi(R, s) e^{st}\, ds. \tag{3.25}$$

In this case

$$\varphi_1 = \frac{R^*}{\rho_0} \frac{1}{2\pi i} \int_l \frac{p^*(s)}{s} e^{-\frac{1}{a_0} s (R - R^*) + st}\, ds, \tag{3.26}$$

where the contour l can be any straight line parallel to the imaginary axis when $\sigma > 0$.

But $p(t) = p^* e^{-t/\theta}$ in accordance with (3.7). Consequently,

$$p(s) = p^* \int_0^\infty e^{-\frac{t}{\theta}} e^{-st}\, dt = p^* \left\{ -\frac{1}{\frac{1}{\theta} + s} e^{-\left(\frac{1}{\theta} + s\right) t} \Big|_0^\infty \right\} =$$
$$= p^* \frac{1}{\frac{1}{\theta} + s}. \tag{3.27}$$

Thus

$$\varphi_1(R, t) = \frac{R^*}{\rho} \frac{1}{2\pi i} \int_l e^{-\frac{1}{a_0} s \left(R - R^*\right) + st} p^* \frac{1}{\left(\frac{1}{\theta} + s\right) s}\, ds, \tag{3.28}$$

- 180 -

since

$$p = \rho_0 \frac{\partial \varphi}{\partial t} = \rho_0 \frac{1}{R} \frac{\partial \varphi_1}{\partial t}, \tag{3.29}$$

so that according to (3.28) and (3.29) we have

$$p(R, t) = p^* \frac{R^*}{R} \frac{1}{2\pi i} \frac{\partial}{\partial t} \int_l e^{s\left(t - \frac{R - R^*}{a_0}\right)} \frac{1}{\left(\frac{1}{\theta} + s\right)s} ds =$$

$$= p^* \frac{R^*}{R} \frac{1}{2\pi i} \int_l e^{s\left(t - \frac{R - R^*}{a_0}\right)} \frac{1}{\frac{1}{\theta} + s} ds. \tag{3.30}$$

In the case when $t < (R - R^*)/a_0$ the integrand tends to zero in the region to the right of l as $s \to \infty$.

At the same time, the function $p(R, t)$ has no singularities in the right half plane. Consequently, when $t < (R - R^*)/a_0$, we have

$$p(R, t) = 0.$$

To the contrary, if $t > (R - R^*)/a_0$, then the integrand tends to zero as $s \to -\infty$ ($s = \sigma + i\tau$, $\sigma > 0$). Then the function $p(R, t)$ has in the left half plane a pole at the point $s = -1/\theta$. On the basis of the residue theorem, we have

$$p(R, t) = p^* \frac{R^*}{R} \frac{1}{2\pi i} 2\pi i e^{-\frac{1}{\theta}\left(t - \frac{R - R^*}{a_0}\right)}.$$

Thus,

$$p(R, t) = \begin{cases} 0 & \text{for } t < \frac{R - R^*}{a_0} \\ p^* \frac{R^*}{R} e^{-\frac{1}{\theta}\left(t - \frac{R - R^*}{a_0}\right)} & \text{for } t \geqslant \frac{R - R^*}{a_0} \end{cases}$$

or, introducing a unit discontinuity function of zero order,

$$p(R, t) = p^* \frac{R^*}{R} e^{-\frac{1}{\theta}\left(t - \frac{R - R^*}{a_0}\right)} \sigma_0\left(t - \frac{R - R^*}{a_0}\right). \tag{3.31}$$

The solution (3.31) determines the direct wave.

On the free surface of the liquid there should be satisfied the boundary condition

$$\rho_0 \frac{\partial \varphi}{\partial t}\Big|_{z=0} = p\Big|_{z=0} = 0. \tag{3.32}$$

This condition can be readily satisfied if, using the principle of linear superposition, we add to the solution (3.31) the term

$$-p' = -p^* \frac{R^*}{R'} e^{-\frac{1}{4}\left(t - \frac{R'-R^*}{a_*}\right)} q_0\left(t - \frac{R'-R^*}{a_0}\right), \qquad (3.33)$$

where, unlike $R = \sqrt{L^2 + (H-z)^2}$, we have $R' = \sqrt{L^2 + (H+z)^2}$.

Thus, the resultant pressure will be

$$p = p^* \frac{R^*}{R} e^{-\frac{1}{4}\left(t - \frac{R-R^*}{a_*}\right)} q_0\left(t - \frac{R-R^*}{a_0}\right) -$$

$$- p^* \frac{R^*}{R'} e^{-\frac{1}{4}\left(t - \frac{R'-R^*}{a_*}\right)} q_0\left(t - \frac{R'-R^*}{a_0}\right). \qquad (3.34)$$

The result obtained admits of a simple and illustrative geometrical interpretation, known as the scheme of mirror reflection of the source and of the sink (Fig. 52).

In this case the direct wave can be regarded as a perturbation due to a source in the lower half plane, while the reflected wave is due to an imaginary sink in the upper half plane. In either case the change in pressure on the fronts of the wave systems is discontinuous.

A formal consequence of Relation (3.34) is the possibility of occurrence of negative stresses in the liquid. The question thus arises whether water can withstand such stresses. Individual experiments have shown that under certain conditions water free of mechanical admixtures can withstand negative pressures up to 280 atm.* However, in practice, one observes most frequently a rapid development of cavitation phenomena and the pressure drops to the vacuummetric value.

The reflection of shock waves from the free surface can be well recorded by experimental methods. The instant of arrival of the reflected wave at a point corresponds to a cut in the pressure curve on the oscillogram of the explosion. By way of illustration, several such oscillograms are shown in Fig. 53. The solid lines here show the pressure oscillograms obtained by experiment, while the dashed curves

correspond to calculations by Formula (3.34). As can be readily seen, there is complete agreement between the theoretical and experimental data up to the instant of arrival of the rarefaction wave. Beyond this the calculation curves indicate a possible existence of a region of negative stresses, which were not recorded experimentally.
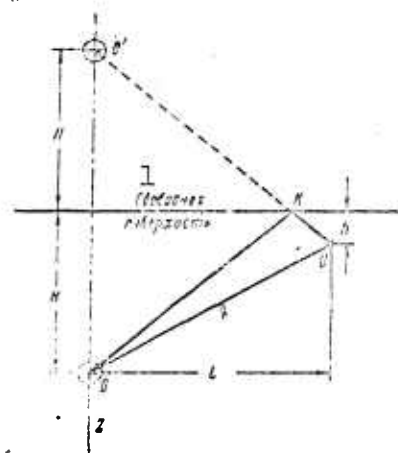
Fig. 52. Diagram of mirror reflection of a source and a sink. 1) Free surface.

The time of action of the positive pressure phase is obviously determined by the difference between the times of arrival of the direct wave and the wave reflected from the free surface at the given point.

Since the distance covered by the direct wave is equal to $\sqrt{L^2 + (H-h)^2}$, and that covered by the reflected wave is $\sqrt{L^2 + (H+h)^2}$, we have

$$t_{ak} = \frac{1}{a_0}\left\{\sqrt{L^2 + (H+h)^2} - \sqrt{L^2 + (H-h)^2}\right\}. \qquad (3.35)$$

If the distance L is large compared with the depth of the charge H and of the measurement point h, then we can assume

$$t_{ak} = \frac{L}{a_0}\left\{\sqrt{1 + \left(\frac{H+h}{L}\right)^2} - \sqrt{1 + \left(\frac{H-h}{L}\right)^2}\right\} \simeq$$

$$\simeq \frac{L}{a_0}\left\{1 + \frac{1}{2}\left(\frac{H+h}{L}\right)^2 - 1 - \frac{1}{2}\left(\frac{H-h}{L}\right)^2\right\}, \qquad (3.36)$$

$$t_{ak} \simeq \frac{2Hh}{La_0}. \qquad (3.37)$$

It follows therefore that the time of action of the positive pressure phase, with allowance for the influence of the free surface, is determined in the acoustic approximation by the purely geometrical characteristics and does not depend on the weight of the charge.

Using as a basis an exponential dependence of the pressure on the time, we obtain for the pressure impulse
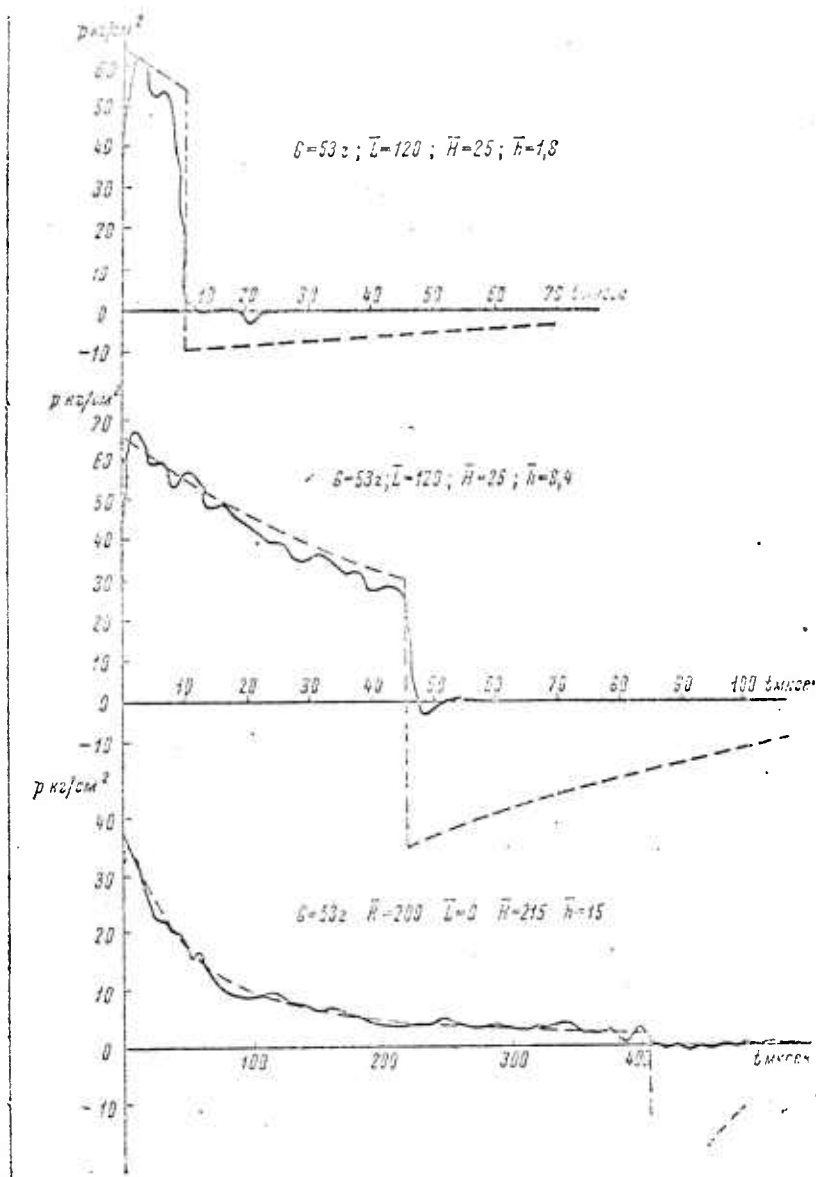
Fig. 53. Characteristic pressure oscillograms with allowance for the influence of the free surface.

$$I_{ak} = \int_0^{t_{ak}} p_m c^{-\frac{t}{\theta}} = p_m \theta \left(1 - e^{-\frac{t_{ak}}{\theta}} \right).\qquad(3.38)$$

Usually the ratio t_{ak}/θ is small, so that with accuracy up to 2% we can assume:

for $t_{ak}/\theta < 0.35$

$$I_{ak} = p_m t_{ak} \left(1 - \frac{t_{ak}}{2\theta} \right);\qquad(3.39)$$

for $t_{aк}/\theta < 0.04$

$$I_{aк} = p_m t_{aк}.$$ (3.40)

The pressure on the front of a shock wave can be calculated by the empirical formula of R. Koul:

$$p_m = \frac{14\,700}{\bar{L}^{1.13}}.$$ (2.151)

The scheme proposed for taking into account the influence of the free surface of the liquid in the acoustic approximation, while simple and clear, has nevertheless a limited region of practical application. If the pressure fields are considered near the free surface, it becomes necessary to take into account the nonlinearity of the reflection process. Then the pressure on the front of the wave turns out to be smaller than in an unbounded liquid, and the time of action of the positive pressure phase is smaller than calculated in the acoustic approximation. The shape of the pressure vs. time curve also changes.

These facts can be explained as follows:

1. Part of the explosion energy goes over into the air if the depth of the charge is low. In the region of the front of an underwater shock wave, the energy becomes redistributed and this results in a reduction of the pressure. Such a process exerts an appreciable influence on the pressure field if the depth of the charge does not exceed one radius.

2. When a shock wave strikes a free surface, a rarefaction wave is produced. According to Zemplen's theorem, the rarefaction wave cannot have a discontinuous front. It is an aggregate of characteristics (elementary pressure-reduction waves), propagating with the local velocity of sound.

Since the local velocity of sound behind the front of the shock wave, added to the velocity of motion of the liquid behind it, is al-

ways larger than the velocity of front displacement, some of the characteristics catch up with the shock wave and attenuate it. To the contrary, one group of characteristics with appreciable negative amplitude has a velocity close to the velocity of sound in the unperturbed medium, and consequently reaches the point of observation later than the shock wave arrival at the same point $(N > a_0)$. Thus, on the one hand, the rarefaction wave decreases the pressure on the front of the direct shock wave and distorts the pressure-time pattern, and on the other hand the time of action of the positive phase of the pressure becomes longer.

It is important to ascertain the range in which the influence of the described nonlinear effects becomes manifest, and to provide a quantitative estimate for it.

We begin the exposition of the nonlinear theory of shock-wave reflection with a determination of the limits of possible application of the acoustic-approximation formulas.

Example 1. Construct the pattern of the pressures at a point located at a distance L = 100 m from the center of explosion of a TNT charge weighing 300 kg. The depth of the charge is assumed to be H = = 10 m, and the depth of the measurement point is h = 2.0 m.

Also calculate the value of the impulse of the pressures in the specified point.

Solution. The radius of the equivalent spherical charge is

$$R_{03} = 0.053 \sqrt[3]{\overline{G}} = 0.053 \sqrt[3]{100} = 0.355 \ \text{m}.$$

The relative distances are

$$\overline{L} = \frac{100}{0.355} = 282; \quad \overline{H} = \frac{10}{0.355} = 28.2; \quad \overline{h} = \frac{2.0}{0.355} = 5.64.$$

The pressure on the front of the shock wave is determined from the formula

$$\dot{p}_m = \frac{14\,700}{L^{1,13}} = \frac{14\,700}{282^{1,13}} = 25{,}2\,\text{kg}/\text{cm}^2.$$

The exponential damping constant is in accordance with Formula (2.156)

$$\theta = 1{,}4\,\frac{R_{03}}{a_0}\,\bar{r}^{0,24} = 1{,}4\,\frac{0{,}355}{1460}\cdot 3{,}877 = 0{,}00132\ \text{sec}.$$

The time interval between the arrivals of the direct and reflected waves at the point is

$$t_{\text{ak}} = \frac{1}{a_0}\left(\sqrt{L^2 + (H+h)^2} - \sqrt{L^2 + (H-h)^2}\right) \simeq \frac{2Hh}{La_0} =$$
$$= \frac{2\cdot 10\cdot 2}{10\cdot 1460} = 273\cdot 10^{-6}\ \text{sec}.$$

The resultant pressure is determined from Formula (3.34)

$$p = p_m e^{-\frac{1}{\theta}\left(t - \frac{R_1}{a_0}\right)}\sigma_0\left(t - \frac{R_1}{a_0}\right) - p_m e^{-\frac{1}{\theta}\left(t - \frac{R_2}{a_0}\right)}\sigma_0\left(t - \frac{R_2}{a_0}\right),$$

or, measuring the time from the instant of arrival of the direct wave at the point, we obtain

$$p = p_m e^{-\frac{1}{\theta}t}\sigma_0(t) - p_m e^{-\frac{1}{t}(t - t_{\text{ak}})}\sigma_0(t - t_{\text{ak}});$$
$$p = 25{,}2\cdot 2{,}72^{-\frac{1}{0,00132}t}\sigma_0(t) - 25{,}2\cdot 2{,}72^{-\frac{1}{t}(t - 0,000273)}\sigma_0(t - 0{,}000273).$$

The pressure-time pattern is shown in Fig. 54. The time is reckoned from the instant of arrival of the direct shock wave at the point.
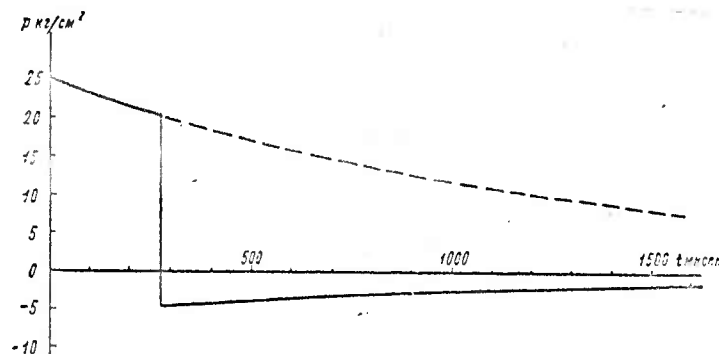
Fig. 54. Illustrating Example 1. Pressure patterns without allowance (dashed) and with allowance (solid line) for the influence of the free surface.

The value of the pressure impulse is determined from Formula (3.3)
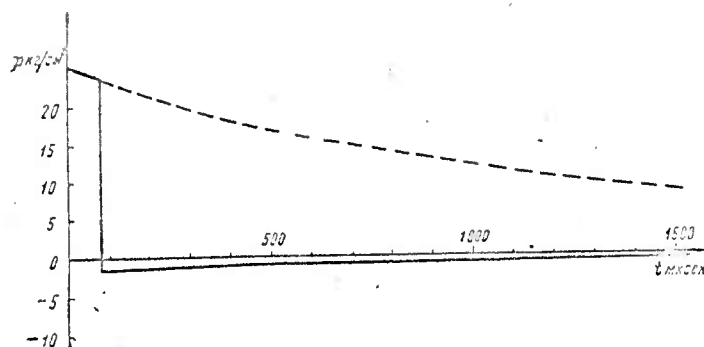
Fig. 55. Illustrating Example 2. Pressure patterns without allowance (dashed) and with allowance (solid line) for the influence of the free surface.

$$I_{\text{ак}} = p_m \theta \left(1 - e^{-\frac{t_{\text{ек}}}{\theta}}\right) = 25{,}2 \cdot 0{,}00132 \left(1 - e^{-\frac{0{,}000273}{0{,}00132}}\right) = 0{,}00622 \ \text{kg-sec/cm}^2.$$

Example 2. Solve the same problem assuming the depth of the charge to be H = 3 m.

Solution. The only difference is that the time interval between the arrivals of the direct and reflected waves at the specified point is different and amounts to in this case

$$t_{\text{ек}} = \frac{1}{a_0}\left(\sqrt{L^2 + (H+h)^2} - \sqrt{L^2 + (H-h)^2}\right) \simeq \frac{2 \cdot H \cdot h}{L \cdot a_0} =$$

$$= \frac{2 \cdot 3 \cdot 2}{100 \cdot 1460} = 0{,}82 \cdot 10^{-4} \ \text{sec}.$$

The pattern-time pressure assumes the form shown in Fig. 55.

The pressure impulse turns out to be equal to

$$I_{\text{ак}} = p_m \theta \left(1 - e^{-\frac{t_{\text{ак}}}{\theta}}\right) = 25{,}2 \cdot 0{,}00132 \left(1 - e^{-\frac{0{,}000082}{0{,}00132}}\right) \simeq 0{,}00200 \ \text{kg-sec/cm}^2.$$

§3. LIMITS OF POSSIBLE APPLICATION OF THE ACOUSTIC APPROXIMATION FORMULAS. ACCOUNT OF THE INFLUENCE OF THE FREE SURFACE IN THE NONLINEAR FORMULATION OF THE PROBLEM

A study of these effects has attracted the interest of many researchers. The main results in this field belong to A.A. Grib, A.G. Ryabin, S.A. Khristianovich, B.V. Zamyshlyayev, and Ya.F. Sharov, who have developed a nonlinear theory of interaction between an underwater shock wave and the free surface of a liquid, and who have proposed ap-

proximate formulas for the estimate of the shock-wave parameters.

We shall adhere below essentially to the exposition of Grib, Khristianovich, and Ryabinin.*

We consider first the simplest case of a plane shock wave of unit amplitude normally incident on the free surface.

The elementary pressure-reduction waves resulting from the reflection will propagate in this case in a medium with constant parameters (p_f, v_f, a_f).

As is well known, such a motion is described by the Riemann singular solution (see §7, Chapter 1). Using the coordinate system of Fig. 56 for the "inverse wave of one direction," we have

$$v + \frac{2}{n-1} a = \text{const.} \qquad (3.41)$$

It is easy to determine the constant in (3.41) by recognizing that on the forward boundary of the rarefaction wave (zero-amplitude characteristic) we have

$$p = p_\phi, \quad \rho = \rho_\phi,$$
$$a = a_\phi, \quad v = v_\phi.$$

Consequently,

$$v + \frac{2}{n-1} a = v_\phi + \frac{2}{n-1} a_\phi,$$

hence

$$v = v_\phi + \frac{2}{n-1} (a_\phi - a). \qquad (3.42)$$

According to the linearized conditions of dynamic compatibility, we have

$$N_\phi = a_0 \left(1 + \frac{n+1}{4Bn} p_\phi\right); \qquad (1.205)$$

$$a_\phi = a_0 \left(1 + \frac{n-1}{2Bn} p_\phi\right); \qquad (1.206)$$

$$v_\phi = a_0 \frac{p_\phi}{nB}, \qquad (1.207)$$

and, as can be readily verified, an expression of the type (1.206) is

valid not only for the wavefront, but also for an arbitrary region behind the front.

Indeed, on the basis of the equation of state of water

$$\frac{p+B}{\rho^n} = \frac{p_0 + B}{\rho_0^n} \simeq \frac{B}{\rho_0^n},$$

$$a = \sqrt{\frac{dp}{d\rho}} = \sqrt{\frac{Bn}{\rho_0}\left(\frac{\rho}{\rho_0}\right)^{n-1}} = a_0\left(\frac{\rho}{\rho_0}\right)^{\frac{n-1}{2}},$$

or, since

$$\frac{\rho}{\rho_0} \simeq 1 + \frac{p}{nB},$$

we have

$$a = a_0\left(1 + \frac{n-1}{2nB}\,p\right). \qquad (1.206a)$$

Substituting (1.206), (1.206a), and (1.207) in (3.42), we obtain

$$v = v_\Phi + \frac{2}{n-1}\,a_0\left(\frac{n-1}{2Bn}\,p_\Phi - \frac{n-1}{2Bn}\,p\right) =$$

$$= a_0\,\frac{p_\Phi}{nB} + a_0\,\frac{p_\Phi - p}{nB} = a_0\,\frac{2p_\Phi - p}{nB}. \qquad (3.43)$$

On the free surface of the liquid we can assume the pressure equal to zero. Then

$$v = a_0\,\frac{2p_\Phi}{nB}.$$

Fig. 56. Reflection of a shock wave from a free surface in the acoustic approximation and in accordance with the Riemann solution. I) Characteristic of zero amplitude; II) characteristic of maximum amplitude; III) acoustic approximation. 1) Time for the pressure to drop from p_f to 0.

In other words, in exactly the same way as in acoustics, the velocity of the liquid particles on the free surface is twice the velocity of the particles behind the wavefront in an unbounded liquid. The direction of motion of the particles in the rarefaction wave is opposed to the direction of wave travel. Because of this, we obtain for the velocity of propagation of an arbitrary wave characteristic

$$N_{xap} = \frac{dz}{dt} = v - a,$$

or, according to (1.206a) and (3.43)

$$N_{xap} = a_0\,\frac{2p_\Phi - p}{nB} - a_0\left(1 + \frac{n-1}{2nB}\,p\right) =$$

$$= - a_0 \left(1 + \frac{n+1}{2nB} p - \frac{2}{nB} p_\Phi \right). \qquad (3.44)$$

The pressure $\underline{p}$ is made up of the pressure p_f in the direct compression wave and the pressure $p_{v.r}$ in the elementary rarefaction wave

$$p = p_f - p_{v.r}.$$

With this, the pressure in the aggregate of the characteristics forming the rarefaction wave varies continuously from zero (zero-amplitude characteristic, $p_{v.r} = 0$, $p = p_f$) to $p_{v.r} = p_f$ ($p = 0$). In accordance with (3.44), the zero-amplitude characteristic moves with velocity

$$N_{\text{хар}} \big|_{p_{\text{в. p}}=0} = - a_0 \left(1 + \frac{n+1}{2nB} p_\Phi - \frac{2}{nB} p_\Phi \right) =$$
$$= - a_0 \left(1 + \frac{n-3}{2nB} p_\Phi \right).$$

For the maximum-amplitude characteristic we have

$$N_{\text{хар}} \big|_{p_{\text{в. p}}=-p_\Phi} = - a_0 \left(1 - \frac{2}{nB} p_\Phi \right).$$

We note also that according to (3.44) the velocity of displacement of the characteristics of the rarefaction wave depends linearly on their amplitude.

If we assume as the time origin the instant of arrival of the front of the direct wave at the point $z = -h$, then the pattern of propagation of wave disturbances in the liquid can be represented in the following fashion (see Fig. 56).

After a time interval, equal to

$$t_\Phi = \frac{h}{N_\Phi} = \frac{h}{a_0} \cdot \frac{1}{1 + \frac{1+n}{4Bn} p_\Phi} \simeq \frac{h}{a_0} \left(1 - \frac{1+n}{4Bn} p_\Phi \right),$$

the direct shock wave reaches the free surface.

The rarefaction wave comprises a pencil of rays emerging from the point $A(t = h/N_f, z = 0)$, with different velocities.

The zero-amplitude characteristic arrives at the point $z = -h$ at

the instant of time

$$t = t_\phi + \frac{h}{N_{\text{хар}}}\Big|_{p_{\text{и. p}}=0} = \frac{h}{a_0}\left(1 - \frac{1+n}{4Bn}\,p_\phi\right) +$$

$$+ \frac{h}{a_0}\frac{1}{1 + \frac{n-3}{2nB}\,p_\phi} \simeq \frac{h}{a_0}\left\{1 - \frac{1+n}{4Bn}\,p_\phi + 1 - \right.$$

$$\left. - \frac{n-3}{2nB}\,p_\phi\right\} = \frac{2h}{a_0}\left(1 - \frac{3n-5}{8Bn}\,p_\phi\right), \qquad (3.45)$$

while the maximum-amplitude characteristic ($p_{v.r} = -p_f$) arrives at a time

$$t = t_\phi + \frac{h}{N_{\text{хар}}}\Big|_{p_{\text{и. p}}=-p_\phi} = \frac{h}{a_0}\left(1 - \frac{1-n}{4Bn}\,p_\phi\right) +$$

$$+ \frac{h}{a_0}\frac{1}{1 - \frac{2}{nB}\,p_\phi} \simeq \frac{2h}{a_0}\left(1 + \frac{7-n}{8Bn}\,p_\phi\right). \qquad (3.46)$$

In the acoustic approximation the compression and rarefaction waves propagate with identical velocities. The time intervals between the arrivals of the direct and reflected waves at the point z = −h is

$$t_{\text{ак}} = \frac{2h}{a_0},$$

and coincides practically with the time calculated by Formula (3.46).*
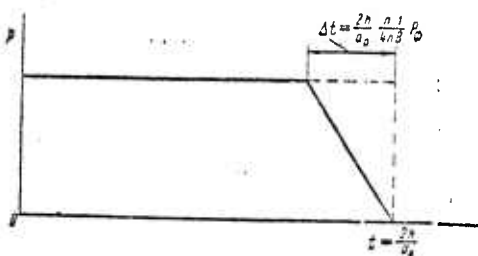
Fig. 57. Character of time variation of pressure in the acoustic approximation and in accordance with the Riemann solution.

Unlike this flow, the Riemann flow is characterized by gradual change in the pressure during a time interval (Fig. 57)

$$\Delta t = \frac{2h}{a_0}\,p_\phi\left[\frac{7-n}{8Bn} + \frac{3n-5}{8Bn}\right] = \frac{2h}{a_0}\,\frac{n+1}{4Bn}\,p_\phi.$$

The ratio

$$\frac{\Delta t}{t_{\text{ак}}} = \frac{n+1}{4Bn}\,p_\phi$$

amounts to less than 1% for pressures $p_f = 100$ atm.

It follows from the foregoing that when a shock wave is normally incident on a free surface, the exact and acoustic solutions are close to each other.
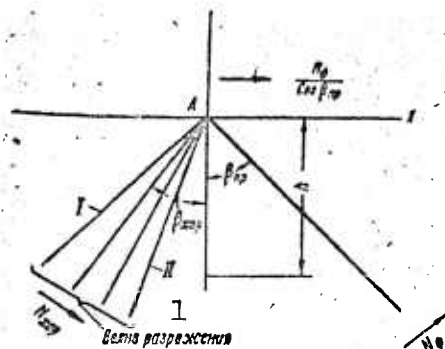
Fig. 58. Diagram showing reflection of shock wave from the free surface of a liquid. I) Maximum-amplitude characteristic; II) zero-amplitude characteristic. 1) Rarefaction wave.

The situation is somewhat different in the case of incidence of a shock wave on a free surface at an arbitrary angle. It is necessary to distinguish here between the regular and irregular reflections.

Let us consider consider first regular reflection (Fig. 58).

The rarefaction wave moves in the region of a liquid perturbed by the passage of the direct wave. Its displacement velocity is made up of the local velocity of sound $\underline{a}$, the projection of the velocity of particle motion behind the front of the direct wave on the direction of displacement of the characteristics

$$v_\phi \cos(\beta_{np} + \beta_{x.p})$$

and the velocity increment due to the motion of the characteristic itself

$$\Delta v = -a_0 \frac{p_\phi - p}{nB}.$$

Thus

$$N_{x \cdot p} = a + v_\phi \cos(\beta_{np} + \beta_{x \cdot p}) - a_0 \frac{p_\phi - p}{nB},$$

or, recognizing that β_{pr} and β_{khar} are close to each other, and discarding terms of second order of smallness,

$$N_{x \cdot p} = a + v_\phi \cos 2\beta_{np} - a_0 \frac{p_\phi - p}{nB}.$$

Substituting into this expression the values of the velocity of sound and the particle velocity, in accordance with (1.206a) and (1.207), we have

$$N_{x \cdot p} = a_0 \left(1 + \frac{n-1}{2Bn} p\right) + a_0 \frac{p_\phi}{nB} \cos 2\beta_{np} - a_0 \frac{p_\phi - p}{nB} =$$

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$$= a_0 \left[1 + \frac{n+1}{2Bn} p - (1 - \cos 2\beta_{np}) \frac{p_\phi}{nB} \right]. \tag{3.47}$$

On the free surface, the direct wave and the characteristics of the rarefaction wave have a common point A (Fig. 58). Consequently,

$$\frac{N_{xap}}{\cos \beta_{xap}} = \frac{N_\phi}{\cos \beta_{np}}. \tag{3.48}$$

In accordance with (3.47) and (3.48)

$$\cos \beta_{xap} = \cos \beta_{np} \frac{N_{xap}}{N_\phi} = \cos \beta_{np} \frac{a_0 \left[1 + \frac{n+1}{2nB} p - \left(1 - \cos 2\beta_{np} \frac{p_\phi}{nB} \right) \right]}{a_0 \left[1 + \frac{n+1}{4nB} p_\phi \right]} =$$

$$= \cos \beta_{np} \left[1 + \frac{n+1}{2nB} p - \left(\frac{n+5}{4} - \cos 2\beta_{np} \right) \frac{p_\phi}{nB} \right]. \tag{3.49}$$

This formula was first obtained in the work of Grib, Ryabinin, and Khristianovich.

At small angles of incidence

$$\cos \beta_{np} \approx 1 - \frac{1}{2} \beta_{np}^2,$$

$$\cos 2\beta_{np} \simeq 1 - 2\beta_{np}^2,$$

and the last equation assumes the form

$$\cos \beta_{xap} = \left(1 - \frac{1}{2} \beta_{np}^2 \right) \frac{1 + \frac{n+1}{2nB} p - 2\beta^2 \frac{p_\phi}{nB}}{1 + \frac{n+1}{4nB} p_\phi},$$

or, discarding small terms of third order,

$$\cos \beta_{xap} = \frac{1 + \frac{n+1}{2nB} p}{1 + \frac{n+1}{4nB} p_\phi + \frac{1}{2} \beta_{np}^2}. \tag{3.50}$$

If the pressures are not too high, then in first approximation

$$\cos \beta_{xap} \simeq 1 - \frac{1}{2} \beta_{np}^2,$$

hence

$$\beta_{xap} \simeq \beta_{np}.$$

Because of this, the time of action of the shock wave in the case of regular reflection differs insignificantly from the time of the acoustic approximation.

The pattern pressure at an arbitrary point of the liquid can be

constructed by starting from the following considerations. Assume that we are considering, as before, some point located at a depth $\underline{h}$ away from the free surface. At the instant when the front of the direct wave arrives at this point $(t = 0)$ the pressure jumps abruptly to a value p_f and remains constant until the zero-amplitude characteristic arrives. From simple geometric constructions (see Fig. 58) it follows that this time interval is

$$t_0 = \frac{h \cos \beta_{np}}{N_\phi} (\mathrm{tg}\,\beta_{np} + \mathrm{tg}\,\beta_{xnp}\,|_{p_{n,\,p}=0});$$

the time of arrival of a characteristic of arbitrary amplitude at the point under consideration is

$$t = \frac{h \cos \beta_{np}}{N_\phi} (\mathrm{tg}\,\beta_{np} + \mathrm{tg}\,\beta_{xnp}),$$

or

$$t = \frac{h}{N_\phi} \sin \beta_{np} \left(1 + \frac{\mathrm{tg}\,\beta_{xnp}}{\mathrm{tg}\,\beta_{np}}\right). \tag{3.51}$$

On the basis of (3.49), neglecting small quantities of second order, we can easily reduce the ratio $\tan \beta_{khar} / \tan \beta_{pr}$ to the form

$$\frac{\mathrm{tg}\,\beta_{xnp}}{\mathrm{tg}\,\beta_{np}} = 1 + \frac{1}{\sin^2 \beta_{np}} \left[\left(\frac{n+5}{4} - \cos 2\beta_{np}\right)\frac{p_\phi}{nB} - \frac{n+1}{4nB} p\right].$$

Substituting this result togetherwith the expression for N in Formula (3.51), we obtain ultimately

$$t = \frac{2h}{a_0} \sin \beta_{np} \left\{ 1 - \frac{n+1}{4nB} p_\phi + \frac{1}{\sin^2 \beta_{np}} \left[\left(\frac{n+5}{4} - \right.\right.\right.$$
$$\left.\left.\left. - \cos 2\beta_{np}\right)\frac{p_\phi}{2nB} - \frac{n+1}{4nB} p\right]\right\}. \tag{3.52}$$

Therefore the time of arrival of the zero-amplitude characteristic $(p_{v.r} = 0,\ p = p_f)$ at the point will be

$$t_0 = \frac{2h}{a_0} \sin \beta_{np} \left\{ 1 - \left[\frac{n+1}{4} + \frac{1}{2\sin^2 \beta_{np}} \left(\frac{n-3}{4} + \right.\right.\right.$$
$$\left.\left.\left. + \cos 2\beta_{np}\right)\right]\frac{p_\phi}{nB}\right\}.$$

For the maximum-amplitude characteristic we obtain

$$t_1 = \frac{2h}{a_0} \sin \beta_{np} \left\{ 1 - \left[\frac{n+1}{4} - \frac{1}{2 \sin^2 \beta_{np}} \left(\frac{n+5}{4} - \right. \right. \right.$$
$$\left. \left. \left. - \cos 2\beta_{np} \right) \right] \frac{p_\phi}{nB} \right\}.$$

These laws enable us to make more precise the previously formu-
lated deduction that is possible to estimate the time of action of a
shock wave in the regular-reflection region with the aid of the acous-
tic equations. Indeed, for example, let the conditions of the problem
be such that an error of about 10% is permissible in the determination
of the time of action of the wave, i.e.,

$$\frac{t_1 - t_0}{t_1} \leqslant 0,1.$$

After substituting the corresponding values of t_1 and t_0, we ob-
tain

$$\frac{n+1}{4nB} \frac{p_\phi}{\sin^2 \beta_{np}} \leqslant 0,1$$

and

$$\beta_{np} \geqslant \arcsin \sqrt{10 \frac{n+1}{4nB} p_\phi}.$$

The considerations presented above are valid not for the entire
possible range of variation of the angles of incidence of the direct
wave. Starting with an angle which we shall henceforth call critical
($\beta*$),* the rarefaction wave will catch up with the front of the direct
wave, distort its form, and change the amplitude. Irregular reflection
sets in.

Let us find the value of the critical angle $\beta*$. To this end, we
compare the velocity of displacement of the front of pressure-
reduction wave (zero-amplitude characteristic) in the direction of the
free surface with the velocity of the front of the direct wave.

The zero-amplitude characteristic moves in the direction of the
free surface with velocity

$$a = a_0 \cos \beta_{np}.$$

The velocity of displacement of the direct wave front in the same direction is

$$\frac{N_{d}}{\cos \beta_{pr}}.$$

The value of the critical angle is determined by the equation

$$a + v_\phi \cos \beta^* = \frac{N_\phi}{\cos \beta^*}.$$

Assuming β^* to be small, and consequently putting $\cos \beta^* = 1 - \beta^{*2}/2$, and using Relations (1.205), (1.206a), and (1.207), we obtain

$$a_0 \left(1 + \frac{n-1}{2Bn} p_\phi\right) + a_0 \frac{p_\phi}{nB}\left(1 - \frac{1}{2} \beta^*\right) = \frac{a_0\left(1 + \frac{1+n}{4Bn} p_\phi\right)}{1 - \frac{1}{2}\beta^*}.$$

Alternately, discarding small quantities of order higher than the second,

$$1 + \frac{n+1}{2Bn} p_\phi = 1 + \frac{1+n}{4Bn} p_\phi + \frac{1}{2}\beta^*,$$

hence

$$\beta^* = \sqrt{\frac{n+1}{2Bn} p_\phi}. \qquad (3.53)$$

If the angle of incidence is $\beta_{pr} > \beta^*$, the pressure-reduction waves do not catch up with the front of the shock wave. Regular reflection occurs, similar to that considered above.

When $\beta_{pr} < \beta^*$ the reflection becomes irregular. Then the common point of the rarefaction-wave and shock-wave fronts moves downward from the free surface. The section of the shock-wave front adjacent to the free surface becomes curved. The pressures on this section become smaller than in an unbounded liquid. The boundary of the irregular-reflection region is shown in Fig. 59, and a diagram for such a reflection is shown in Fig. 60.

Let us find the point of intersection B of the front of the shock wave and the front of the pressure-reduction wave. Below this point, the front of the shock wave does not bend, and the pressure on it is
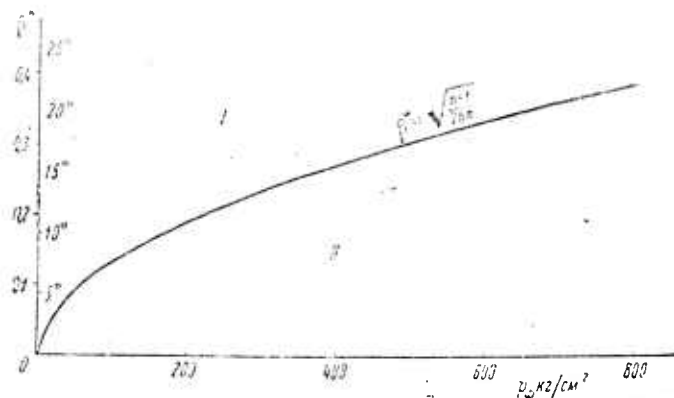
Fig. 59. Regions of regular (I) and irregular (II) reflection.
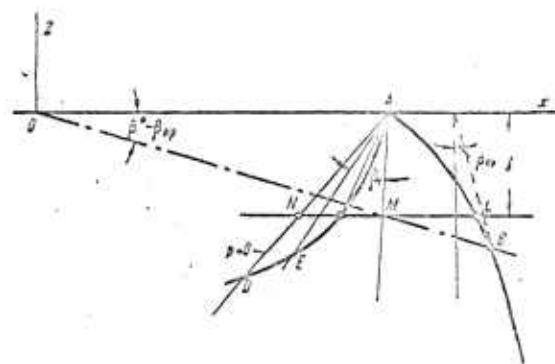
Fig. 60. Diagram showing the bending of the shock-wave front by the rarefaction-wave characteristics.

the same as in an unbounded liquid. We assume a coordinate system such as shown in Fig. 60. The equation of the front of the pressure-reduction wave, assuming approximately that the velocity behind the front of the shock wave is directed parallel to the free surface, will be*

$$(x - v_\Phi t)^2 + z^2 = a^2 t^2,$$

or, according to (1.206) and (1.207),

$$\left(x - a_0 t \frac{p_\Phi}{Bn}\right)^2 + z^2 = a_0^2 t^2 \left[1 + \frac{n-1}{2} \frac{p_\Phi}{Bn}\right]^2. \tag{3.54}$$

We put

$$x = a_0 t [1 + X]; \quad z = a_0 t Z, \tag{3.55}$$

and then Eq. (3.54) assumes the form

$$\left[1 + X - \frac{p_\phi}{Bn}\right]^2 + Z^2 = \left[1 + \frac{n-1}{2}\frac{p_\phi}{Bn}\right]^2.$$

Near the free surface the coordinate $\underline{x}$ is close to $a_0 t$, and consequently X is small. Discarding quantities of second order of smallness, we obtain from the last equation

$$X + \frac{Z^2}{2} = \frac{n+1}{2}\frac{p_\phi}{Bn} = \beta^{*\prime}. \tag{3.56}$$

The equation of the front of the direct wave, assuming that $\tan \beta_{pr} \simeq \beta_{pr}$, will be

$$x + \beta_{np} z = \frac{N_\phi}{\cos \beta_{np}},$$

or

$$x + \beta_{np} z = a_0 t \left[1 + \frac{n+1}{4}\frac{p_\phi}{Bn} + \frac{\beta_{np}^2}{2}\right].$$

Changing over to the variables X and Z and recognizing that $(n + 1)p_f/4Bn = \beta^{*2}/2$, we obtain

$$X + \beta_{np} Z = \frac{1}{2}\left(\beta^{*\prime} + \beta_{np}^2\right). \tag{3.57}$$

The coordinates of the point B where the front of the shock wave intersects the front of the pressure-reduction wave is determined from a simultaneous solution of Eqs. (3.56) and (3.57).

After elementary calculations we obtain*

$$Z_B = -(\beta^* - \beta_{np}),$$
$$X_B = \frac{\beta^{*\prime} + \beta_{np}^2}{2} + \beta_{np}(\beta^* - \beta_{np}).$$

The point B moves along a straight line passing through the point O and inclined at a small angle $\beta^* - \beta_{pr}$ to the free surface.

We determine in the same way the position of the front of the pressure-reduction wave with amplitude equal to $\underline{p}$. The velocity of motion of such a front will differ from the velocity of sound by an amount $-a_0(p_f - p)/nB$. Consequently,

$$\left(x - a_0 t \frac{p_\phi}{Bn}\right)^2 + z^2 = a_0^2 t^2 \left[1 + \frac{n-1}{2nB}p - \frac{p_\phi - p}{nB}\right]^2,$$

or

$$\left(x - a_0 t \frac{p_\phi}{Bn}\right)^2 + z^2 = a_0^2 t^2 \left[1 + \frac{n+1}{2nB} p - \frac{p_\phi}{nB}\right]^2.$$

Making a change of variables and retaining only first-order quantities, we obtain

$$X + \frac{Z^2}{2} = \frac{n+1}{2} \frac{p}{Bn}. \tag{3.56a}$$

Let us find the equation of the curved wavefront. Assume that it is characterized by the relation

$$X = X(Z).$$

Previously we had

$$x = a_0 t [1 + X],$$

or

$$x = a_0 t [1 + X(Z)].$$

The velocity of the points of the front parallel to the free surface will be

$$\frac{dx}{dt} = a_0 [1 + X] + a_0 t \frac{dX}{dZ} \frac{dZ}{dt} = a_0 \left[1 + X - Z \frac{dX}{dZ}\right].$$

On the other hand, this velocity is equal to

$$\frac{N}{\cos a} = a_0 \left[1 + \frac{n+1}{4} \frac{p}{Bn} + \frac{\beta^2}{2}\right].$$

Comparing the expressions obtained and taking into consideration the fact that $\beta = -dX/dZ$, we obtain

$$X - Z \frac{dX}{dZ} = \frac{n+1}{4} \frac{p}{Bn} + \frac{1}{2} \left(\frac{dX}{dZ}\right)^2. \tag{3.58}$$

Eliminating the pressure from (3.56a) and (3.58) we obtain the differential equation of the wavefront

$$\left(\frac{dX}{dZ}\right)^2 + 2Z \frac{dX}{dZ} + \frac{Z^2}{2} - X = 0.$$

This equation can be integrated in closed form.* We have

$$\left(\frac{dX}{dZ}\right)^2 + 2Z \frac{dX}{dZ} + Z^2 = X + \frac{Z^2}{2},$$

or

$$\frac{dX}{dZ} + Z = \sqrt{X + \frac{Z^2}{2}}.$$

Putting $X = Z^2 v$, we arrive at an equation with separable variables

$$\frac{dZ}{Z} = \frac{dv}{\sqrt{v + \frac{1}{2}} - 1 - 2v}.$$

Its integral, returning to the variables X and Z, will be

$$Z = \frac{C}{\sqrt{\frac{X}{Z^2} + \frac{1}{2}} - \frac{1}{2}},$$

or

$$X = C + CZ - \frac{Z^2}{4}.$$

The constant C can be determined from the condition that the integral curve must pass through the point B:

$$X_B = \frac{\beta^{*2} + \beta_{np}^2}{2} + \beta_{np}(\beta^* - \beta_{np}),$$
$$Z_B = -(\beta^* - \beta_{np}).$$

After simple calculations we obtain

$$C = -\frac{\beta^* + \beta_{np}}{2}.$$

Consequently,

$$X = \left[\frac{\beta^* + \beta_{np}}{2}\right] - \frac{\beta^* + \beta_{np}}{2} Z - \frac{Z^2}{4},$$

or

$$X + \frac{Z^2}{2} = \left[\frac{\beta^* + \beta_{np}}{2} - \frac{Z}{2}\right]^2.$$

But, according to (3.56a),

$$X + \frac{Z^2}{2} = \frac{n+1}{2}\frac{p}{Bn}.$$

Thus

$$\left[\frac{\beta^* + \beta_{np}}{2} - \frac{Z}{2}\right]^2 = \frac{n+1}{2}\frac{p}{Bn}. \tag{3.59}$$

By definition

$$\beta^{*} = \frac{n+1}{2} \frac{p_{\Phi}}{Bn}. \tag{3.53}$$

Comparing (3.59) and (3.53) we arrive at the equation

$$\frac{p}{p_{\Phi}} = \frac{1}{4} \left[1 + \frac{\beta_{np}}{\beta^{*}} - \frac{Z}{\beta^{*}} \right]^{2}. \tag{3.60}$$

At the point B we have $Z_{B} = -(\beta^{*} - \beta_{pr})$ and the pressure turns out to be the same as in an unbounded liquid, $p = p_{\infty}$.

On the free surface $(Z = 0)$ we have

$$\frac{p}{p_{\Phi}} \Big|_{z=0} = \frac{p'}{p_{\Phi}} = \frac{1}{4} \left[1 + \frac{\beta_{np}}{\beta^{*}} \right]^{2}. \tag{3.61}$$

Relation (3.60) can be reduced after simple transformations to the form

$$p = p' \left[1 + \frac{\beta^{*} - \beta_{np}}{\beta^{*} + \beta_{np}} \frac{Z}{Z_{B}} \right]^{2}. \tag{3.62}$$

Inasmuch as $Z/Z_{B} = h/h_{B}$, it follows from (3.62) that the pressure on the curved portion of the front changes from a value $p' = p_{f}/4(1 + \beta_{pr}/\beta^{*})$ to $p = p_{f}$ parabolically.

Let us find now the time of action of the shock wave.

To this end we consider the propagation of wavefronts with pressure $p < p'$, produced near the points of the free surface of the liquid, such waves move along the free surface more slowly than the point A; their envelope AE is shown in Fig. 60.

The angle β_{1} between the line AE and the perpendicular to the free surface is determined by the equation

$$\cos \beta_{1} = \frac{a + u}{a' + u'},$$

or, according to (1.206a) and (1.207)

$$\cos \beta_{1} \simeq 1 - \frac{1}{2} \beta_{1}^{2} = \frac{1 + \frac{n-1}{2nB} p + \frac{1}{nB} p}{1 + \frac{n-1}{2nB} p' + \frac{1}{nB} p'},$$

hence

$$\beta_{1} = \sqrt{\frac{n+1}{Bn} (p' - p)}.$$

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The point E on the end of the line segment AE is located on the front of a wave with pressure p, pertaining to the pressure-reduction wave originating at the point O.

Its coordinates will be

$$Z_E = -\sqrt{\frac{n+1}{Bn}(p'-p)}$$

and according to (3.56a)

$$X_E = -\frac{Z_E^2}{2} + \frac{n+1}{2}\frac{p}{Bn}.$$

The geometric locus of the points E corresponding to different pressures p is the parabola

$$X + Z^2 = \frac{n+1}{2Bn}p' = \frac{1}{4}(\beta^* + \beta_{np})^2. \tag{3.63}$$

This parabola terminates at the point D, which corresponds to the boundary of the wave with pressure $p = 0$.

The coordinates of the point D are determined by the equations

$$Z_D = -\frac{\beta^* + \beta_{np}}{\sqrt{2}},$$

$$X_D = -\frac{(\beta^* + \beta_{np})^2}{4}.$$

From the point A there emerge at different slopes the rarefaction-wave characteristics. Their equation is

$$X - Z_n = \frac{1}{4}(\beta^* + \beta_{np})^2. \tag{3.64}$$

In particular, there passes through the point D a characteristic

$$X - Z\frac{\beta^* + \beta_{np}}{\sqrt{2}} = \frac{1}{4}(\beta^* + \beta_{np})^2, \tag{3.65}$$

on which the pressure is equal to zero.

The relations established enable us to determine the time of action of the shock wave. Assume that the measurement point is located at a depth h from the free surface. The maximum pressure at this point occurs on the surface of the wave and in accordance with (3.62) its value is (Fig. 60)

$$p_L = p' \left[1 + \frac{\beta^* - \beta_{np}}{\beta^* + \beta_{np}} \frac{Z}{Z_B} \right]^2$$

at

$$X_L = \left[\frac{\beta^* + \beta_{np}}{2} \right]^2 - \frac{\beta^* + \beta_{np}}{2} Z - \frac{Z^2}{4}.$$

The point M lies on the parabola (3.63).
Consequently,

$$X_M = \frac{(\beta^* + \beta_{np})^2}{4} - Z^2.$$

The value of the pressure at this point is calculated with the aid of Relation (3.56a)

$$X + \frac{Z^2}{2} = \frac{n+1}{2} \frac{p}{Bn},$$

$$\frac{(\beta^* + \beta_{np})^2}{4} - \frac{Z^2}{2} = \beta^{*2} \frac{p_M}{p_\phi},$$

$$p_M = p_\phi \frac{1}{\beta^{*2}} \left\{ \frac{(\beta^* + \beta_{np})^2}{4} - \frac{Z^2}{2} \right\}.$$

Finally, the pressure becomes equal to zero on the line AD.
According to (3.65)

$$X_N = \frac{1}{4} (\beta^* + \beta_{np})^2 + Z \frac{\beta^* + \beta_{np}}{V^2}.$$

If the depth of the measurement point $\underline{h}$ is such that $|Z| < |Z_B|$ and $|Z| < |Z_D|$, the time of action of the wave will obviously be

$$\tau = \frac{X_L - X_M}{a_0} = \frac{a_0 t (x_L - x_M)}{a_0} =$$

$$= -\frac{Z}{t} \left\{ (\beta^* + \beta_{np}) \left(\frac{1}{2} + \frac{1}{V^2} \right) + \frac{Z}{4} \right\} =$$

$$= \frac{h}{a_0} \left[\left(\frac{1}{2} + \frac{1}{V^2} \right) (\beta^* + \beta_{np}) - \frac{h}{4R} \right],$$

where $R = a_0 t$.

The maximum value of h/R does not exceed $\beta^* - \beta_{pr}$ (see Fig. 60).
Consequently, discarding the term h/4R, we can approximately put

$$\tau = 1,2 \frac{h}{a_0} (\beta^* + \beta_{np}). \tag{3.66}$$

At large depths of the measurement point, the time of action of the wave differs little from the value obtained in the acoustic approx-

imation.

All the considerations advanced above are valid only for the case of propagation of a plane wave.

The situation becomes much more complicated in the case of an explosion of a spherical charge near the free surface. Here the pressure on the wavefront does not remain constant, and the angle of encounter between the wave and the surface also changes. The formulas obtained can then serve only for qualitative estimates. Nonetheless, their use yields a result that is closer to reality than the acoustic formulas.

In view of this, let us make a few remarks concerning the estimate of the influence of the free surface in the case of the explosion of a spherical charge.

The pressure on the front of the wave in an unbounded liquid is estimated in this case by the formula

$$p_\phi = \frac{14\,700}{r^{1.13}}. \tag{2.151}$$

At a large distance from the center of the explosion we have

$$\beta_{np} \simeq \frac{H}{r},$$

where H is the depth of the charge.

The critical angle of encounter between the shock wave and the free surface is, in accordance with (3.53)

$$\beta^* = \sqrt[7]{\frac{n+1}{2Bn} \frac{14\,700}{r^{1.13}}} \simeq 1.66 \frac{1}{r^{0.56}}, \tag{3.67}$$

since

$$\beta^* = \frac{H^*}{r} = \frac{\overline{H}^*}{r},$$

$$\overline{H}^* = 1.66 r^{0.44}. \tag{3.68}$$

At relative depths of the charge $\overline{H} > \overline{H}^*$, the reflection is regular. The pressure on the front of the shock wave is the same as in an unbounded liquid.

When $\overline{H} < \overline{H}^*$ it is necessary to take into account the nonlinear effects due to the influence of the free surface of the liquid on the pressure fields in an underwater explosion. As a result of the propagation of the rarefaction wave, tensile stresses and cavitation zones can then arise in the liquid, as will be discussed in greater detail in the next section.

Example. Plot a pressure-time pattern at a specified point with allowance for the nonlinear influence of the free surfaces, taking as the initial data those of the first and second problems of the preceding section:

$$\text{a)} \quad G = 300 \text{ kg}, \quad L = 100 \text{ m},$$
$$h = 2.0 \text{ m}, \quad H = 10 \text{ m};$$
$$\text{b)} \quad G = 300 \text{ kg}, \quad L = 100 \text{ m},$$
$$h = 2.0 \text{ m}, \quad H = 3 \text{ m}.$$

Compare these results.

Solution: a) First we consider the first case. The angle of encounter β_{pr} between the direct wave and the free surface is

$$\beta_{пр} = \frac{H}{L} = \frac{10}{100} = 0.1.$$

The critical angle is determined from the formula (3.53):

$$\beta^* = \sqrt{\frac{n+1}{2Bn} \, p_\Phi},$$

where

$$p_\Phi = \frac{14\,700}{Z^{1.13}} = \frac{14\,700}{282.13} = 25.2 \text{ kg/cm}^2.$$

Thus

$$\beta^* = \sqrt{\frac{8.15 \cdot 25.2}{2 \cdot 3015 \cdot 7.15}} = 0.0386.$$

Since $\beta_{pr} > \beta^*$, the reflection is regular.

Up to the instant of arrival of the zero-amplitude characteristic at the point, the pressure pattern remains the same as in an unbounded

·liquid.

According to (3.52), the time when a rarefaction wave of given amplitude arrives at the point is

$$t_\text{p} = \frac{2h}{a_0} \sin\beta_\text{пр} \left\{ 1 - \frac{n+1}{4Bn} p_\phi + \right.$$

$$\left. + \frac{1}{\sin^2\beta_\text{пр}} \left[\left(\frac{n+5}{4} - \cos 2\beta_\text{пр} \right) \frac{p_\phi}{2Bn} - \frac{n+1}{4Bn} p_\phi \right] \right\}.$$

The time of arrival of the zero-amplitude characteristic ($p_\text{v.r} = 0$; $p = p_f$) at the point will be

$$t_0 = \frac{2h}{a_0} \sin\beta_\text{пр} \left\{ 1 - \left[\frac{n+1}{4} + \frac{1}{2\sin^2\beta_\text{пр}} \left(\frac{n-3}{4} + \cos 2\beta_\text{пр} \right) \right] \frac{p_\phi}{nB} \right\}.$$

or after substituting the corresponding values

$$p_m = 25.2 \ \text{кг/см}^2; \quad \sin\beta_\text{пр} = 0.0998; \quad \cos\beta_\text{пр} = 0.995; \quad \cos 2\beta_\text{пр} = 0.980.$$

$$t_0 = \frac{2\cdot 2}{1460} \, 0.0998 \left\{ 1 - \left[\frac{7.15+1}{4} + \frac{1}{2(0.0998)^2} \left(\frac{7.15-3}{4} + 0.980 \right) \right] \frac{25.2}{7.15\cdot 3045} \right\} =$$
$$= 241\cdot 10^{-6} \ \text{sec}.$$

At the instant t_0 the pressure in the direct wave will be

$$p\big|_{t=t_0} = p_m c^{-\frac{t_0}{\theta}} = 25.2\cdot e^{-\frac{0.000241}{0.00132}} = 21.0 \ \text{kg/cm}^2.$$

The rarefaction wave with amplitude $p\big|_{t=t_0}$

$$(p_\text{v.r} = -21.0 \ \text{kg/cm}^2; \quad p = 25.2 - 21.0 = 4.2 \ \text{kg/cm}^2),$$

will arrive at the specified point at the instant of time

$$t = \frac{2\cdot 2.0}{1460} \, 0.0998 \left\{ 1 - \frac{7.15+1}{4\cdot 7.15\cdot 3045} \, 25.2 + \right.$$
$$\left. + \frac{1}{(0.0998)^2} \left[\left(\frac{7.15+5}{4} - 0.980 \right) \frac{25.2}{2\cdot 7.15\cdot 3045} - \frac{8.15}{4\cdot 7.15\cdot 3045} \, 4.2 \right] \right\} =$$
$$= 296\cdot 10^{-6} \ \text{sec}.$$

We can assume approximately that at this instant of time the pressure in the shock wave is zero.

Comparison of the results of the calculation with the data of the first example of the preceding section is made in Fig. 61.

It is easy to see that in the region of irregular reflection it is perfectly permissible to use the acoustic-approximation formulas.

b) Now let H = 3 m.
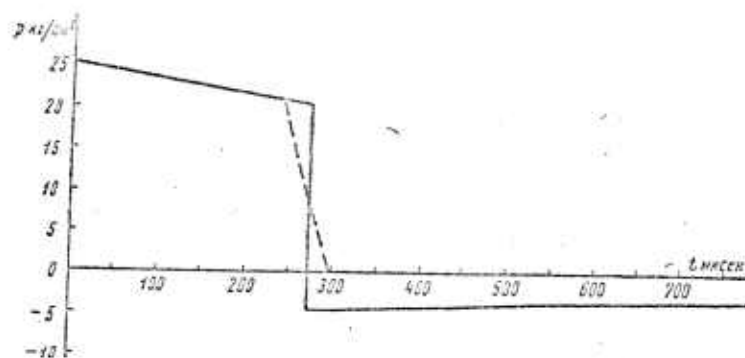
Fig. 61. Comparison of the acoustic solution (solid line) and the solution in which the nonlinear effect of the free surface is taken into account (regular reflection, dashed).

The angle of encounter between the direct wave and the free surface turns out to be

$$\beta_{np} = \frac{H}{L} = \frac{3}{100} = 0,03.$$

The critical angle is as before $\beta^* = 0.0686$.

Since $\beta_{pr} < \beta^*$, irregular reflection occurs.

The pressure on the front of the shock wave is determined from Formula (3.62)

$$p = p' \left[1 + \frac{\beta^* - \beta_{np}}{\beta^* + \beta_{np}} \frac{Z}{Z_B} \right]^2,$$

where

$$p' = p_m \frac{1}{4} \left[1 + \frac{\beta_{np}}{\beta^*} \right]^2,$$

$$\frac{Z}{Z_B} = \frac{h}{h_B}, \quad \text{"but"} h_B = L (\beta^* - \beta_{np}).$$

Thus

$$p = p_m \frac{1}{4} \left[1 + \frac{\beta_{np}}{\beta^*} \right]^2 \left[1 + \frac{h}{L (\beta^* + \beta_{np})} \right]^2.$$

Substituting the corresponding values of the quantities, we obtain

$$p = 25,2 \frac{1}{4} [1 + 0,437]^2 \left[1 + \frac{2}{0,0986 \cdot 100} \right]^2 = 18,8 \text{ kg/cm}^2.$$

The time of action of the positive phase of the shock wave will
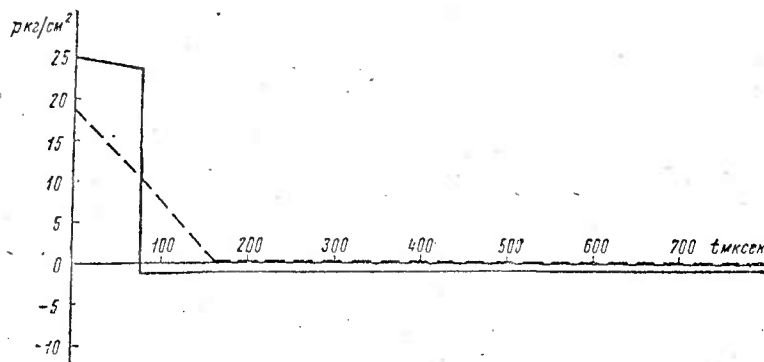
Fig. 62. Comparison of the acoustic solu-
tion (solid line) with the solution
in which the nonlinear influence of the
free surface is taken into account (irreg-
ular reflection, dashed).

be in accordance with (3.66)

$$\tau = 1{,}2\,\frac{h}{a_0}\,(\beta^* + \beta_{np}) = 1{,}2\,\frac{2\cdot 0{,}0986}{1460} = 162\cdot 10^{-6}\,\text{sec.}$$

The pressure pattern is closely approximated by a linear relation-
ship. Figure 62 shows a comparison of the results of the present cal-
culations with the calculations in the acoustic approximation (Example
2 of §3).

An essential difference is clearly seen in the final estimates.

It must be borne in mind, however, that for spherical waves the
computation scheme presented is crudely approximate, since the rela-
tions employed are strictly speaking valid only for plane waves.

§4. OCCURRENCE AND DEVELOPMENT OF CAVITATION IN THE REFLECTION OF AN
UNDERWATER SHOCK WAVE FROM A FREE SURFACE

As is well known, cavitation is a process wherein the gas dis-
solved in a liquid is released and a vapor-gas mixture is produced. An
analysis of these phenomena is closely related with a study of the
strength characteristics of a liquid on occurrence of tensile stresses.

It is stated in the papers of Ya.I. Frenkel' and Ya.B. Zel'dovich
that the upper limit of the bulk strength of water is a quantity on
the order of 2000-3000 atm.* Bridge has shown experimentally that dis-

tilled water free of mechanical impurities and air bubbles, can withstand a tensile stress of approximately 280 atm, and its strength depends essentially on the temperature (Fig. 63). It was also clarified that the presence of solid particles, bubbles of dissolved gas, and other similar inclusions in the liquid greatly reduces the ability of the liquid to withstand tensile forces. One can assume, finally, that the very process of loading the liquid is of no little importance in the estimate of its strength.

However, the material available at the present time does not provide an unambiguous answer to the question concerning the values of the tensile stresses that water can withstand upon propagation of a rarefaction wave.
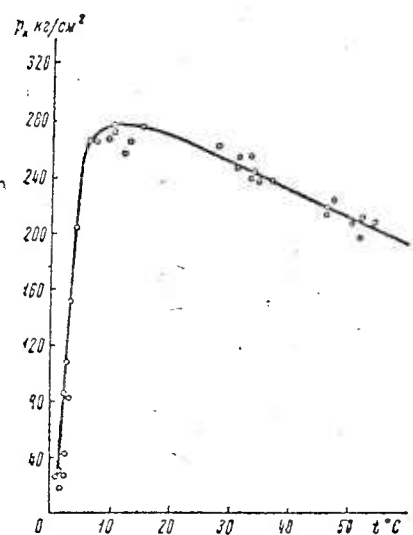
Fig. 63. Cavitation pressure according to Bridge's experimental data.

In order for cavitation to set in, the tensile forces must overcome the hydrostatic-pressure forces and cause a rarefaction equal to the value p_k. Consequently,

$$p_{pe3} = -(|p_k| + p_0), \quad (3.69)$$

where p_{rez} is the resultant pressure causing cavitation of the liquid, and p_0 is the hydrostatic pressure at the specified depth $\underline{h}$

$$p_0 = p_{aтм} + \gamma h. \quad (3.70)$$

Let us consider normal incidence of a plane wave of exponential form on the free surface:

$$p = p_m e^{-\frac{t}{\theta}}. \quad (3.71)$$

We use the scheme of mirror reflection of the source and sink, and obtain without difficulties that the resultant pressure at the
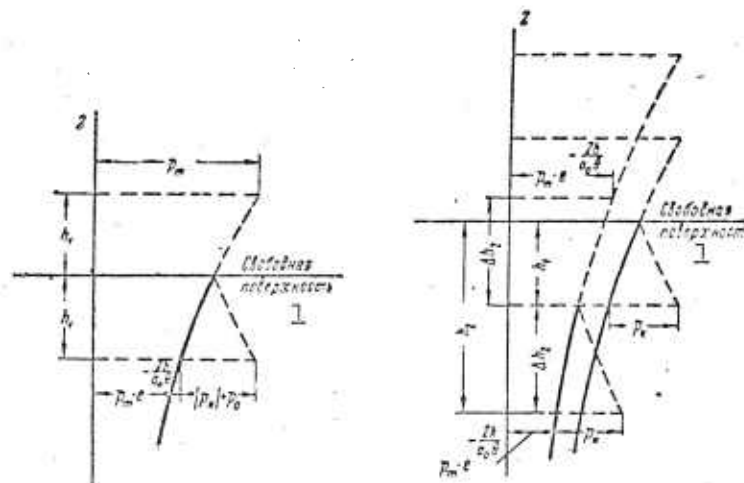
Fig. 64. Formation of cavitational discontinuities. 1) Free surface.

depth $\underline{h}$ amounts to

$$-p_m + p_m e^{-\frac{2h}{a_0 \theta}} = p_m \left(e^{-\frac{2h}{a_0 \theta}} - 1 \right). \qquad (3.72)$$

If the quantity is equated to the pressure causing cavitation in the liquid, we obtain a relation from which we can readily find the thickness h_1 of the cavitational layer (Fig. 64):

$$p_m \left(e^{-\frac{2h_1}{a_0 \theta}} - 1 \right) = - (|p_\kappa| + p_0), \qquad (3.73)$$

hence

$$h_1 = \frac{a_0 \theta}{2} \ln \left\{ 1 - \frac{|p_\kappa| + p_0}{p_m} \right\}. \qquad (3.74)$$

The liquid will move in the cavitational layer in the direction of the free surface. The pressure in it will have a value close to the pressure of saturated water vapor, i.e., close to zero.

For the tail part of the direct wave, the first cavitational layer will play the role of a new free surface of liquid, the only difference being that on the free surface the pressure is equal to atmospheric, while on the boundary of the cavitational discontinuity it is close to zero. Consequently, the tail part of the wave will be reflected from the surface of the first cavitational layer. A rarefac-

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tion wave will arise, the propagation of which can give rise to a new cavitational discontinuity.

The depth h_2 of the surface of the second cavitational discontinuity can obviously be determined from the equation (Fig. 64):

$$p_m\left(e^{-\frac{2h_1}{a_0\theta}} - e^{-\frac{2h_1}{a_0\theta}}\right) = -|p_\kappa|, \qquad (3.75)$$

hence

$$e^{-\frac{2h_1}{a_0\theta}} = -\frac{|p_\kappa|}{p_m} + e^{-\frac{2h_1}{a_0\theta}},$$

but

$$e^{-\frac{2h_1}{a_0\theta}} = 1 - \frac{|p_\kappa| + p_0}{p_m}.$$

Consequently,

$$h_1 = -\frac{a_0\theta}{2}\ln\left\{1 - \frac{l|p_\kappa| + p_0}{p_m}\right\}. \qquad (3.76)$$

The process of formation of cavitational layers will continue until negative stresses in excess of $|p_k|$ will arise in the water.

The depth of the i-th cavitational layer will in this case be

$$h_2 = -\frac{a_0\theta}{2}\ln\left\{1 - \frac{2|p_\kappa| + p_0}{p_m}\right\}. \qquad (3.77)$$

Its thickness is

$$\Delta h_i = h_i - h_{i+1} = -\frac{a_0\theta}{2}\ln\frac{1 - \frac{l|p_\kappa| + p_0}{p_m}}{1 - \frac{(l-1)|p_\kappa| + p_0}{p_m}}. \qquad (3.78)$$

Perfectly analogous reasoning enables us to find the depth and thickness of the i-th cavitational layer in the case of oblique incidence of a plane wave on the free surface. The only difference is that the initial equation contains as the coefficient of the exponent the sine of the angle between the normal to the front of the wave and the surface of the liquid:

$$p_m\left[e^{-\frac{2h_1}{a_0\theta}\sin\beta} - 1\right] = -(|p_\kappa| + p_0). \qquad (3.79)$$

As a result

$$h_1 = -\frac{a_0 \vartheta}{2 \sin \beta} \ln\left[1 - \frac{l|p_\kappa| + p_0}{p_m}\right]. \tag{3.80}$$

The distance from the i-th cavitational discontinuity to the surface of the water and the thickness of the i-th cavitational layer will be

$$h_i = -\frac{a_0 \vartheta}{2 \sin \beta} \ln\left\{1 - \frac{l|p_\kappa| + p_0}{p_m}\right\}, \tag{3.81}$$

$$\Delta h_i = -\frac{a_0 \vartheta}{2 \sin \beta} \ln \frac{1 - \frac{l|p_\kappa| + p_0}{p_m}}{1 - \frac{(i-1)|p_\kappa| + p_0}{p_m}}. \tag{3.82}$$

The formed cavitational layers, moving with different velocities at definite instants of time, will collide with one another. As a result of the collision, new systems of shock waves will arise, the propagation of which in the liquid leads to repeated increases in the pressure.

Let us consider this process using as an example the motion of the first cavitational layer. Quite approximately, one can regard this layer as a mass of liquid moving with a certain average velocity under the influence of the forces of gravity and atmospheric counterpressure. Under this assumption, the differential equation of layer motion can be written in the following form:

$$\rho_0 h_1 \ddot{z} = -p_\text{атм} - g\rho_0 h_1, \tag{3.83}$$

where g is the acceleration due to the earth's gravity.

If $z = z_{10}$ and $\dot{z} = \dot{z}_{10}$ when $t = t_1$, then the solution of (3.83) will be

$$z(t) = z_{10} + \dot{z}_{10}(t - t_1) - \frac{p_\text{атм} + g\rho_0 h_1}{2\rho_0 h_1}(t - t_1)^2; \tag{3.84}$$

$$\dot{z}(t) = \dot{z}_{10} - \frac{p_\text{атм} + g\rho_0 h_1}{\rho_0 h_1}(t - t_1). \tag{3.85}$$

The first cavitational layer extends from a depth h_1 to the free surface. Therefore the quantities z_{10} and $\dot{z}_{10}$ can be determined from an analysis of the motion of the points of the free surface. It is ob-

vious that

$$\dot{z}_{10} = \frac{2p(t_1)}{\rho_0 a_0} \sin \beta; \qquad (3.86)$$

$$z_{10} = \int_0^{t_1} \dot{z}_n(t)\, dt = \int_0^{t_1} \frac{2p(t)}{\rho_0 a_0} \sin \beta = \frac{2 \sin \beta}{\rho_0 a_0} I(t_1), \qquad (3.87)$$

where $I(t_1)$ is the impulse of the pressure of the direct wave.

The maximum displacement of the cavitational layer can be readily calculated by equating its velocity of motion to zero.

According to (3.85), we obtain

$$t_m - t_1 = \dot{z}_{10} \frac{\rho_0 h_1}{p_{\text{ATM}} + g\rho_0 h}, \qquad (3.88)$$

from which we obtain in accordance with (3.84)

$$z_{\max} = z_{10} + \dot{z}_{10}^2 \frac{\rho_0 h_1}{p_{\text{ATM}} + g\rho_0 h} - \dot{z}_{10}^2 \left(\frac{\rho_0 h_1}{p_{\text{ATM}} + g\rho_0 h} \right)^2 \frac{p_{\text{ATM}} + g\rho_0 h}{2\rho_0 h_1} =$$

$$= z_{10} + \frac{1}{2} \dot{z}_{10}^2 \frac{\rho_0 h_1}{(p_{\text{ATM}} + g\rho_0 h_1)}. \qquad (3.89)$$

Starting with the instant of time t_m, the layer begins to move downward, and at a certain instant it will collide with the main mass of liquid.

Let us examine this process. To this end, we first find the displacement of the liquid at a depth h_1 at the instant of formation of the cavitational discontinuity:

$$z_{20} = \int_0^{2t_1 = \frac{2h_1}{a_0} \sin \beta} \frac{p_m}{\rho_0 a_0} \cdot \sin \beta\, e^{-\frac{t}{\theta}}\, dt = \frac{p_m \theta}{\rho_0 a_0} \sin \beta \left(1 - e^{-\frac{2h_1}{a_0 \theta} \sin \beta} \right) =$$

$$= \frac{p_m \theta}{\rho_0 a_0} \sin \beta \frac{|p_k| + p_0}{p_m} = \theta \frac{|p_k| + p_0}{\rho_0 a_0} \sin \beta. \qquad (3.90)$$

From then on the vertical component of the velocity of the particles on the boundary will be*

$$\dot{z}_2 = \left[\frac{p_0}{\rho_0 a_0} + \frac{2p(t_0 - h_1)}{\rho_0 a_0} \right] \sin \beta a_0 (t - t_1) =$$

$$= \frac{1}{\rho_0 a_0} \sin \beta \left[p_0 + 2p_m e^{-\frac{t + \frac{h_1}{a_0} \sin \beta}{\theta}} \right] a_0 (t - t_1). \qquad (3.91)$$

Thus, the displacement of the water particles on the boundary of the cavitational discontinuity turns out to be

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$$z_2(t) = z_{20} + \int_{t_1}^{t} \dot{z}_2 \, dt = z_{20} + \frac{1}{\rho_0 a_0} \sin \beta \left\{ p_0 (t - t_1) + \right.$$

$$\left. + 2 p_m \eta e^{-\frac{t_1 + \frac{h_1}{a_0} \sin \beta}{\eta}} \left(1 - e^{-\frac{t - t_1}{\eta}} \right) \right\} \sigma_0 (t - t_1), \qquad (3.92)$$

or, inasmuch as $t_1 = (h_1/a_0) \sin \beta$, and

$$e^{-\frac{2h_1}{a_0 \eta} \sin \beta} = 1 - \frac{|p_k| + p_0}{p_m},$$

we have

$$z_2(t) - z_{20} = 2 \frac{p_m \eta}{\rho_0 a_0} \sin \beta \left\{ \frac{p_0}{2 p_m \eta} (t - t_1) + \right.$$

$$\left. + \left(1 - \frac{|p_k| + p_0}{p_m} \right) \times \left(1 - e^{-\frac{t - t_1}{\eta}} \right) \right\} \sigma (t - t_1). \qquad (3.93)$$

The instant of collision between the upper cavitational layer and the main mass of the liquid can be determined from the equation

$$z_{(t = t_{\text{соуд}})} - z_{10} = z_{2(t = t_{\text{соуд}})} - z_{20}, \qquad (3.94)$$

where the quantities contained in (3.94) are determined by Eqs. (3.84), (3.87), (3.90), and (3.93).

Upon encounter between the cavitational layer and the main mass of the liquid, secondary compression waves are produced and propagate in opposite directions from the collision surface. The amplitude of these waves can be readily determined from the condition that the normal velocity components must be equal:

$$\dot{z}_{1(t_{\text{соуд}})} + \frac{p_{\text{соуд}}}{\rho_0 a_0} \sin \beta = \dot{z}_{2(t_{\text{соуд}})} - \frac{p_{\text{соуд}}}{\rho_0 a_0} \sin \beta, \qquad (3.95)$$

hence

$$p_{\text{соуд}} = \frac{\rho_0 a_0}{2 \sin \beta} \left[\dot{z}_{2(t_{\text{соуд}})} - \dot{z}_{1(t_{\text{соуд}})} \right]. \qquad (3.96)$$

It is possible to consider in similar fashion the process of collision between the remaining cavitational layers.

An approximate estimate of the character of motion of the cavitational layers can be obtained by applying the laws of momentum and energy conservation. Indeed, estimating, for example, the initial momen-

tum of the two upper layers, we can state that it is equal to the impulse of the pressure forces in the direct wave prior to the instant of formation of the second cavitational discontinuity:

$$k_{20} = I(2t_2).$$ (3.97)

From then on the system of the two layers is acted upon only by the forces of atmospheric counterpressure and gravity, with the latter being negligible.

Thus, the momentum acquired by the layers will decrease in the course of time by an amount equal to the impulse of the atmospheric forces $p_{atm}(t - t_2)$:

$$k_{20} - k_2(t) = p_{\text{атм}}(t - t_2).$$ (3.98)

After collision between the first and second cavitational layers, their average velocity of motion will be

$$\dot{z}_1(t) = \frac{k_2(t)}{\rho_0 h_2} = \frac{I(2t_2) - p_{\text{атм}}(t - t_2)}{\rho_0 h_2}.$$ (3.99)

The kinetic energy of motion is expressed by the well-known relationship

$$T_2(t) = \frac{\rho_0 h_2}{2} \dot{z}_1^2(t) = \frac{1}{2\rho_0 h_2} [I(2t_2) - p_{\text{атм}}(t - t_2)]^2.$$ (3.100)

It is obvious that this quantity should be equal to the energy of the shock wave from which the work done by the counterpressure forces is subtracted.

Since the waves propagate at an angle α to the free surface, the initial value of the energy will be $E(2t_2) \sin \beta$.

The work done by the counterpressure forces is approximately equal to

$$A = p_{\text{атм}} [z_1(t) - z_2(t_2)],$$ (3.101)

where

$$z_2(t_2) \simeq \frac{I(2t_2)}{\rho_0 a_0} \sin \beta.$$

Thus,

$$E(2t_2) - p_{\text{атм}}[z_1(t) - z_2(t_2)] = T_2(t). \qquad (3.102)$$

On the basis of (3.102) and (3.100) we establish the following connection between the resultant displacement of the layers, on the one hand, and the momentum and energy of the shock wave on the other:

$$z_1(\dot{t}) = z_2(t_2) + \frac{E(2t_2) - T_2(t)}{p_{\text{атм}}} = \frac{I(2t_2)}{\rho_0 a_0}\sin\beta +$$
$$+ \frac{E(2t_2)}{p_{\text{атм}}}\sin\beta - \frac{1}{2\rho_0 h_2 p_{\text{атм}}}[I(2t_2) - p_{\text{атм}}(t - t_2)]^2, \qquad (3.103)$$

or, inasmuch as

$$h_2 = \frac{a_0 t_2}{\sin\beta}, \qquad (3.104)$$

we have

$$z_1(t) = \sin\beta\left\{\frac{I(2t_2)}{\rho_0 a_0} + \frac{E(2t_2)}{p_{\text{атм}}} - \frac{[I(2t_2) - p_{\text{атм}}(t - t_2)]^2}{2\rho_0 a_0 t_2 p_{\text{атм}}}\right\}. \qquad (3.105)$$

After a definite time interval, the two upper layers are overtaken by the third cavitational layer, and at the instant of time t_k, k upper layers catch up with the k + 1-st layer.

Arguing in identical fashion as before, we can readily obtain the following general expressions characterizing the motion of the k joining layers:

$$\dot{z}(t) = \frac{I(2t_k) - p_{\text{атм}}(t - t_k)}{\rho_0 t_k} = \sin\beta\left[\frac{I(2t_k) - p_{\text{атм}}(t - t_k)}{\rho_0 a_0 t_k}\right]; \qquad (3.106)$$

$$z(t) = \sin\beta\left\{\frac{I(2t_k)}{\rho_0 a_0} + \frac{E(2t_k)}{p_{\text{атм}}} - \frac{[I(2t_k) - p_{\text{атм}}(t - t_k)]^2}{2\rho_0 a_0 p_{\text{атм}} t_k}\right\}. \qquad (3.107)$$

We recall that the values of the impulse of the pressures and of the energy in the shock wave are determined in terms of the shock wave parameters as follows:

$$I(2t_k) = p_m \theta\left(1 - e^{-\frac{2t_k}{\theta}}\right), \qquad (3.108)$$

or

$$I(2t_k) = p_m \theta\frac{k|p_k| + p_0}{p_m} = \theta(k|p_k| + p_0); \qquad (3.109)$$

$$E(2t_k) = \frac{p_m^2{}^0}{2\rho_0 a_0}\left\{1 - e^{-\frac{t_k}{\tau}}\right\} =$$

$$= \frac{p_m^2{}^0}{2\rho_0 a_0}\left\{1 - \left[1 - \frac{k|p_k| + p_0}{p_m}\right]^2\right\}. \qquad (3.110)$$

Formulas (3.106)-(3.110) enable us to calculate the maximum displacement of the point of the free surface of the liquid.

Assuming that by the instant of time corresponding to z_{max} there have joined together N upper layers, we obtain on the basis of (3.106)

$$t_{max} = t_N + \frac{I(2t_N)}{p_{\text{атм}}}. \qquad (3.111)$$

Consequently,

$$z_{max} = \sin\beta\left[\frac{I(2t_N)}{\rho_0 a_0} + \frac{E(2t_N)}{p_{\text{атм}}}\right]. \qquad (3.112)$$

If the value of the maximum displacement of the points of the free surface is estimated without account of the cavitational phenomena, it is obvious that

$$z_{1\,max} = \int_0^\infty \dot{z}_n(t)\,dt = \sin\beta\int_0^\infty \frac{2p(t)}{\rho_0 a_0}\,dt = \sin\beta\frac{2I}{\rho_0 a_0}. \qquad (3.113)$$

On comparison of (3.112) and (3.113) we see how large the changes introduced by the account of the cavitation processes in the final relations are.

The calculations performed show that the maximum amplitudes of the compression waves arising upon collision between the cavitational layers and the main mass of the liquid reach two-thirds of the pressure on the front of the direct wave:

$$p_{\text{соуд}} \simeq \frac{2}{3}p_m. \qquad (3.114)$$

The duration of the positive phase of these waves amounts to

$$\tau_{\text{соуд}} \simeq 1,8\theta. \qquad (3.115)$$

The instant of collision is approximately equal to

$$t_{\text{соуд}} \simeq \frac{2p_m\theta}{p_{\text{атм}}}. \qquad (3.116)$$
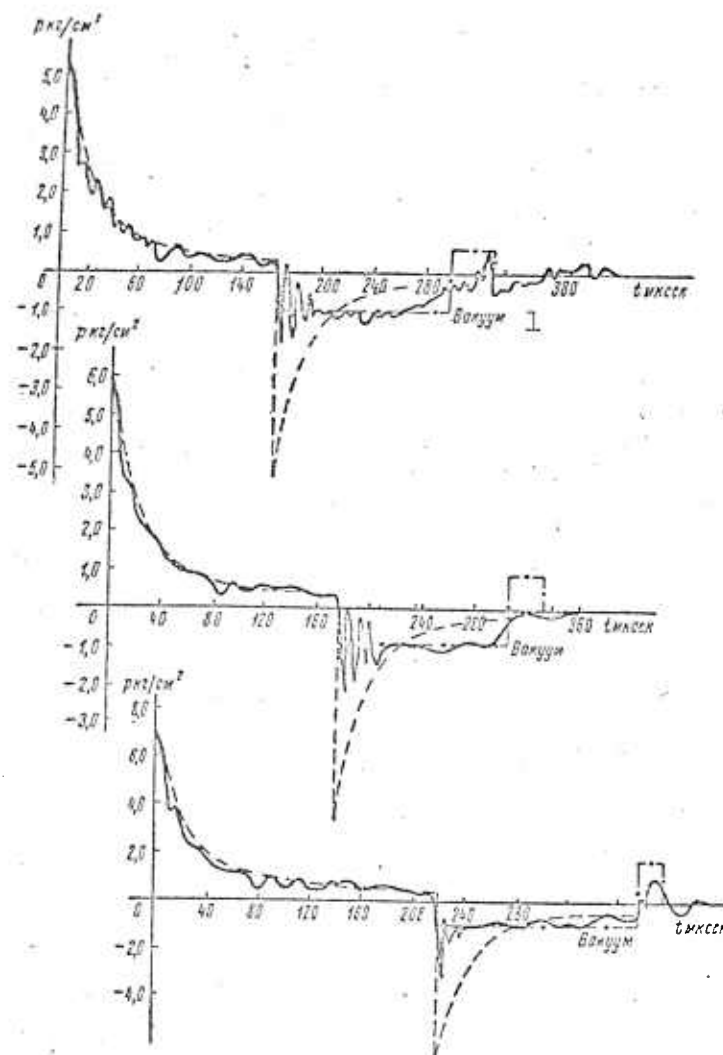
Fig. 65. Pressure oscillograms in the cavitation zone: solid line — experimental data; dash-dot — calculation with account of the cavitation; dashed — calculation without account of cavitation. 1) Vacuum.

The foregoing theory, developed by B.V. Zamyshlyayev, greatly schematizes the actual process. It is necessary to determine more accurately the value of the negative stress p_k, at which the region of cavitation begins to form. One can hardly justify fully the assumption of surface cavitational discontinuities and of the displacement of the cavitational layers with certain average velocities.

Calculations performed with the aid of the foregoing formulas are, however, in satisfactory agreement with many experiments, as evidenced by the oscillograms shown in Fig. 65. A more rigorous solution of this

problem was obtained by A. N. Patrashev.

We note also that an account of the cavitational phenomena in the propagation of shock waves in the liquid is essential not so much in the study of reflection from the free surface, but also in the analysis of the interaction between a shock wave and a structure. Some remarks on this question will be made in the last chapter of the book.

§5. QUALITATIVE PATTERN OF REFLECTION OF AN UNDERWATER SHOCK WAVE FROM THE BOTTOM OF A RESERVOIR

A study of the influence of the bottom of a reservoir on the pressure fields produced in an underwater explosion entails difficulties of principal character. It is necessary first of all to know the mechanical properties of the grounds. Yet, the available limited data offer evidence of a rather wide range of acoustic impedance of grounds (from 0.1 to 4.8 of the acoustic impedance of water), thus predetermining a varied character of reflection.* In many cases it is necessary to take into account the inhomogeneity of the grounds and their layered structure. Of great significance in the study of wave propagation is, in addition, the relief of the bottom, the character of which is not amenable as a rule to a mathematical estimate.

In view of the foregoing, it is possible at present to consider the qualitative rather than quantitative aspect of the phenomenon.

For an approximate analysis, we start out with the following assumptions. We consider the surface of the bottom to be a plane interface between two media. The ground will be regarded as an isotropic elastic half space.

As is well known, two types of elastic waves can propagate through unbounded isotropic solids: longitudinal waves, which result from bulk deformation of the material, and transverse waves, due to shear deformation. Sometimes these waves are called dilatation and distortion

waves, respectively. The velocity of propagation of longitudinal (c) and transverse (b) waves is given by the relations

$$c = \sqrt{\frac{\lambda + 2\mu}{\rho}}; \qquad\qquad (3.117)$$

$$b = \sqrt{\frac{\mu}{\rho}}, \qquad\qquad (3.118)$$

where λ and μ are elastic constants, frequently called the Lame constants, and which determine completely the elastic properties of an isotropic body.*

In addition to the indicated two systems of waves, in the presence of an interface between two media another type of wave can also arise, the so-called Rayleigh surface waves. The amplitude of the Rayleigh waves decreases with depth exponentially. Consequently, they propagate as it were in two dimensions. They attenuate with distance more slowly than the longitudinal and transverse waves. The rate of propagation of the surface waves amounts to 0.87-0.96 of the velocity of propagation of the transverse waves.
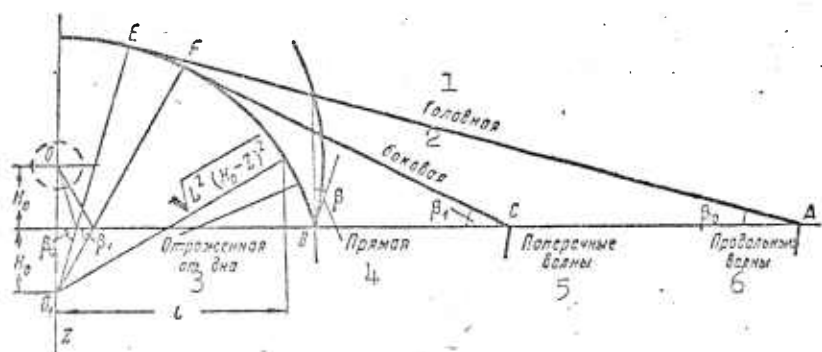
Fig. 66. Diagram showing reflection of a shock wave from the bottom of a reservoir. 1) Frontal; 2) lateral; 3) reflected from bottom; 4) direct; 5) transverse waves; 6) longitudinal waves.

The buildup of compression pressure in seismic waves, unlike shock waves, is not jumpwise but gradual. This is due to the fact that in elastoplastic media the compressibility of the medium increases with increasing pressure over a considerable range of pressures. Since

the velocity of the compression wave is inversely proportional to the compressibility of the medium, the regions of the wave with large pressure will move with lower velocity, and the formation of a steep pressure front is impossible.

One can present the following qualitative description of the wave picture in a liquid upon single reflection from the bottom of a water reservoir. Assume that at some point $z = -H_0$, in the coordinate system indicated in Fig. 66, an explosion occurred and at some instant of time the direct wave comes to the bottom of the water reservoir. Its angle of incidence will be

$$\beta = \arccos \frac{L}{\sqrt{L^2 + H_0^2}}.$$ (3.119)

The rate of displacement of the wave along the surface of the bottom amounts to

$$N_{\text{A}} = \frac{a_0}{\cos \beta}.$$ (3.120)

At first, at large angles β, the velocity N_d will exceed the velocity of propagation of the longitudinal and transverse waves in the ground, $\underline{c}$ and $\underline{b}$. The perturbations in the liquid will be characterized by the direct and reflected waves. The reflection will be regular.

At some value of the angle $\beta = \beta_0$, the velocities N_d and $\underline{c}$ become equal, i.e., the following relation becomes true:

$$\frac{a_0}{\cos \beta_0} = c.$$ (3.121)

The angle β_0 is called the angle of total internal reflection of the longitudinal waves. When $\beta < \beta_0$ there occurs the so-called first irregular reflection. The longitudinal perturbations in the ground overtake the direct shock wave and produce an additional wave disturbance in the liquid. This disturbance is customarily called the frontal wave.

With further propagation of the shock wave, its velocity N_d decreases and becomes at a certain $\beta = \beta_1$ equal to the velocity of the transverse waves in the ground:

$$\frac{a_0}{\cos \beta_1} = b;$$
(3.122)

$$\beta_1 = \arccos \frac{a_0}{b}.$$
(3.123)

The angle β_1 is called the angle of total internal reflection of the transverse waves.

When $\beta < \beta_1$ the transverse waves in the ground overtake the direct shock wave and cause still another wave system in the liquid (lateral wave). The so-called second irregular reflection takes place.

In order to construct the fronts of the lateral and frontal waves, one can use the mirror-reflection scheme. It is easy to verify that the tangents to the circle at the points where it crosses the rays drawn from the center O_1 at angles β_0 and β_1 to the surface of the bottom will be the envelopes of the spherical surfaces of elementary disturbances, which determine the frontal and lateral waves, respectively (Fig. 66).

As was already mentioned earlier, along the surface separating the ground from the water there propagates a Rayleigh wave. The disturbance which it produces in the liquid is customarily called the bottom wave.

Thus, when a shock wave is incident on the bottom of a reservoir there arise in addition to the reflected wave three additional systems of wave disturbances: frontal, lateral, and bottom waves. Each is reflected from the free surface of the water.

The process of successive reflection of waves from the two boundary surfaces continues, generally speaking, without limit. In the majority of cases, however, only the first system of waves is of practical significance.

This is the general outline of the qualitative pattern of reflection of an underwater shock wave from the bottom of a reservoir. We develop below some results of the linear theory, which at the present time is rather fully developed.

§6. PRINCIPAL RESULTS OF THE LINEAR THEORY OF REFLECTION OF AN UNDERWATER SHOCK WAVE FROM THE BOTTOM OF A RESERVOIR

In the linear formulation, the problem of the reflection of a shock wave from the bottom of a reservoir can be expressed as follows.

The specified pressure variation in the upper half space (ideal liquid) on a spherical surface of radius $R = R_0$ is

$$p = p_m e^{-\frac{t}{\theta}}. \tag{3.124}$$

The connection between the pressure and the velocity of the particles in this half space is expressed with the aid of a function of the potential φ, with

$$p = \rho_0 \frac{\partial \varphi}{\partial t}, \tag{3.125}$$

$$\vec{v} = \operatorname{grad} \varphi. \tag{3.126}$$

The strain and stress field in the lower half space (elastic medium) is determined by the system of equations of elasticity theory:

$$\vec{u} = \operatorname{grad} \varphi + \operatorname{rot} \vec{\psi}, \tag{3.127}$$

$$\frac{\partial^2 \varphi}{\partial t^2} = \left[\frac{\lambda + 2\mu}{\rho} \right] \Delta \varphi; \tag{3.128}$$

$$\frac{\partial^2 \vec{\psi}}{\partial t^2} = \frac{\mu}{\rho} \Delta \vec{\psi}, \tag{3.129}$$

where $\vec{u}$ is the vector of the strain velocity of the elastic medium; λ and μ are the elastic constants of the medium (Lame constants)*; φ is the scalar displacement potential; $\vec{\psi}$ is the vector displacement potential.

On the plane interface between the ideal liquid and the elastic medium there is satisfied the rigid-contact condition (the normal components of the particle velocities and of the pressures are equal).

An analysis of this system of equations under the above-formulated boundary conditions was made, using the method of incomplete separation of the variables, by Ye. I. Shemyakin.*

Without stopping to discuss the complicated mathematical apparatus employed by Ye. I. Shemyakin, we present the final results of his investigations.

The pressure in the direct wave is**

$$p = \frac{p_m R_{03}^{1,13}}{[\sqrt{L^2 + (H_1 - h_1)^2}]^{1,13}} e^{-\frac{t-t_n}{\theta}}, \qquad (3.130)$$

where p_m is the value of the pressure on the initial spherical surface R_{03}; H_1 is the distance from the center of the charge to the bottom; h_1 is the distance from the point of observation to the bottom; L is the distance in the horizontal direction; t_n is the time of arrival of the direct wave at the point (L, h_1).

Relation (3.130) can be written, in accordance with (2.151), in the form

$$p = \frac{14\,700}{\bar{r}_1^{1,13}} e^{-\frac{t-t_n}{\theta}}, \qquad (3.131)$$

where

$$\bar{r}_1 = \frac{1}{R_{03}} \sqrt{L^2 + (H_1 - h_1)^2}; \qquad (3.132)$$

$$t_n = \frac{1}{a_0} \sqrt{L^2 + (H_1 - h_1)^2}. \qquad (3.133)$$

So long as the velocity of propagation of the shock wave along the bottom of the reservoir exceeds the velocity of propagation of the longitudinal waves in the ground, $a_0/\cos \beta > c$, the reflection will be regular.

The pressure in the reflected wave is

$$p_{\text{отр}} = A_0 \frac{14\,700}{\bar{r}_2^{1,13}} e^{-\frac{t-t_{\text{отр}}}{\theta}}, \qquad (3.134)$$

in this formula

$$\bar{r}_2 = \frac{1}{R_{03}} \sqrt{L^2 + (H_1 + h_1)^2};$$

(3.135)

$$t_{\text{отр}} = \frac{1}{a_0} \sqrt{L^2 + (H_1 + h_1)^2};$$

(3.136)

A_0 is the reflection coefficient.

When $\beta = 90°$, i.e., when the wave is normally incident on the surface, the coefficient A_0 is calculated from the formula

$$A_0 = \frac{\rho_r c - \rho_0 a_0}{\rho_r c + \rho_0 a_0}.$$

(3.137)

For an analysis of the angles $90° > \beta > \arccos \frac{a_0}{c}$, this coefficient differs little from the quantity calculated by Formula (3.137), and only near $\beta_1 = \arccos \frac{a_0}{c}$ does it change radically and attain a value of unity.

The first irregular reflection occurs at angles of incidence $\beta < \beta_1 = \arccos \frac{a_0}{c}$.

The pressure in the reflected wave in the region of the first irregular reflection is characterized by the dependence

$$p_{\text{отр}} = \frac{14\,700}{\bar{r}_2^{1,13}} \left[A_1 \sigma_0 (t - t_{\text{отр}}) - \right.$$

$$\left. - B_l \overline{E}_l \left(\frac{t - t_{\text{отр}}}{\theta} \right) \sigma_0 (t - t_l) \right] e^{-\frac{t - t_{\text{отр}}}{\theta}},$$

(3.138)

where

$$A_1 = \frac{(2-x)^4 + 16(x-1)\left(1 - \delta_1^2 x\right) - \sigma^2 x^4 \frac{1 - \delta_1^2 x}{\delta_2^2 x - 1}}{(2-x)^4 + \left[4\sqrt{x-1}\sqrt{1 - \delta_1^2 x} + \sigma x^2 \frac{\sqrt{1 - \delta_1^2 x}}{\sqrt{\delta_2^2 x - 1}} \right]^2},$$

(3.139)

$$B_1 = \frac{1}{\pi} \frac{2\sigma x^2 (2-x)^2 \frac{\sqrt{1 - \delta_1^2 x}}{\sqrt{\delta_2^2 x - 1}}}{(2-x)^4 + \left[4\sqrt{x-1}\sqrt{1 - \delta_1^2 x} + \sigma x^2 \frac{\sqrt{1 - \delta_1^2 x}}{\sqrt{\delta_2^2 x - 1}} \right]^2};$$

(3.140)

$$x = \frac{1}{\delta_2^2 \cos^2 \beta}; \quad \sigma = \frac{\rho_0}{\rho_r}; \quad \delta_1 = \frac{b}{c}; \quad \delta_2 = \frac{b}{a_0};$$

t_{otr} is the time of arrival of the reflected wave at the point, calcu-

lated by Formula (3.136); t_1 is the time of arrival of the frontal wave at the point,

$$t_1 = \frac{1}{c}\left(L + \frac{H_1 - h_1}{\delta_1} V\overline{\delta_2^2 - \delta_1^2}\right); \qquad (3.141)$$

and $\overline{E}_i\left(\frac{t - t_{\text{отр}}}{\theta}\right)$ is the Euler function.*

The first term of Eq. (3.138) describes a process similar to the reflection in the regular zone. The second term characterizes the distortion of the reflected-wave pattern as a result of interaction with the ground, which has gone into motion. Its influence begins at the instant when the frontal wave arrives at the point ($t > t_1$).

The section of the pattern of the compression zone in the interval $t_1 < t < t_{\text{otr}}$ is customarily called the "foreshock."

If the velocity of propagation of the transverse waves in the ground, b, exceeds the velocity of sound in the water, a_0, then when $\beta < \beta_2 = \arccos\frac{a_0}{b}$ there sets in the zone of second irregular reflection.

The pressure field in the reflected wave in the zone of second irregular reflection is characterized by a relation analogous to (3.138):

$$p_{\text{отр}} = \frac{14\,700}{r_2^{1,13}}\left[A_2\sigma_0\left(t - t_{\text{отр}}\right) - \right.$$

$$\left. - B_2\overline{E}_i\left(\frac{t - t_{\text{отр}}}{\theta}\right)\sigma_0\left(t - t_1\right)\right]e^{-\frac{t - t_{\text{отр}}}{\theta}}. \qquad (3.142)$$

However, the coefficients A_2 and B_2 have a somewhat different form:

$$A_2 = \frac{\left[(2-x)^2 - 4\sqrt{1-x}\,\sqrt{1-\delta_1^2 x}\,\right]^2 - \sigma^2 x^4\,\frac{1 - \delta_1^2 x}{\delta_2^2 x - 1}}{\left[(2-x)^2 - 4\sqrt{1-x}\,\sqrt{1-\delta_1^2 x}\,\right]^2 + \sigma^2 x^4\,\frac{1 - \delta_1^2 x}{\delta_2^2 x - 1}}; \qquad (3.143)$$

$$B_2 = \frac{1}{\pi}\frac{2\sigma x^2\,\frac{\sqrt{1-\delta_1^2 x}}{\sqrt{\delta_2^2 x - 1}}\left[(2-x)^2 - 4\sqrt{1-x}\,\sqrt{1-\delta_1^2 x}\,\right]}{\left[(2-x)^2 - 4\sqrt{1-x}\,\sqrt{1-\delta_1^2 x}\,\right]^2 + \sigma^2 x^4\,\frac{1 - \delta_1^2 x}{\delta_2^2 x - 1}}. \qquad (3.144)$$

Here the coefficient B_2 already determines the joint influence of

the disturbances due to the longitudinal and transverse ground oscillations.

Let us note now the qualitative laws that result from the formulas given above.

It is obvious, first of all, that the coefficients A_i and B_i are functions of the acoustic properties of the grounds and of the angle of incidence β only. This enables us to construct for the most typical grounds, once and for all, diagrams of $A_i(\beta)$ and $B_i(\beta)$ which are convenient for practical calculations.

Table 16 lists the characteristics of some types of grounds. Figures 67-69 show the angular diagrams for rocky, clay, and sandy bottoms. Frequently the last two types of grounds form only a thin near-bottom layer that lies on a rocky base.

TABLE 16

1 Типы грунтов дна	2 Плотность грунта $\rho_г$, кгсек²/м⁴	3 Скорость распространения продольных волн с, м/сек	4 Скорость распространения поперечных волн b, м/сек	5 Безразмерные параметры грунта		
				$\sigma = \dfrac{\rho_0}{\rho_г}$	$\delta_1 = \dfrac{b}{c}$	$\delta_2 = \dfrac{b}{a_0}$
6 Песчано-илистый	160—190	1400—1700	600— 900	0,640—0,540	0,350—0,640	0,40—0,60
7 Глинистый	220—230	2000—2200	1000—1100	0,465—0,445	0,445—0,550	0,66—0,74
8 Песчаник	250—240	2500—3000	1200—1700	0,410—0,425	0,400—0,680	0,80—1,10
9 Известняк	240—250	3000—3500	1700—1800	0,425—0,410	0,485—0,600	1,10—1,20
10 Гранит	260—270	5000—5500	3000—3100	0,395—0,380	0,570—0,620	2,00—2,05

1) Types of bottom ground; 2) density of ground ρ_g, kg-sec^2/m^4; 3) velocity of propagation of longitudinal waves, c, m/sec; 4) velocity of propagation of transverse waves, b, m/sec; 5) dimensionless parameters of the ground; 6) sand-silt; 7) clay; 8) sandstone; 9) limestone; 10) granite.

If the thickness of such a layer is much smaller than the wavelength, the reflection of the underwater shock wave occurs in exactly the same way as from a solid bottom having the characteristics of the
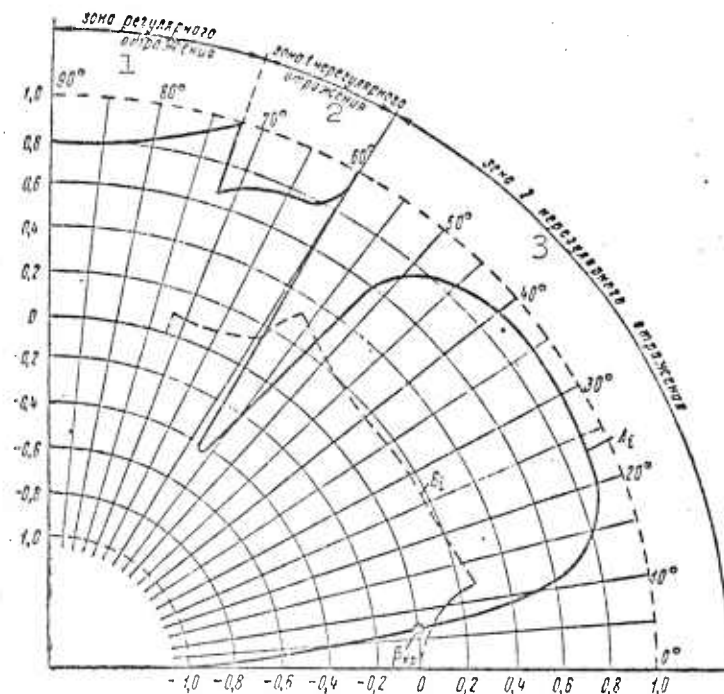
Fig. 67. Angle diagram for a rocky ground.
1) Regular reflection zone; 2) first ir-
regular reflection zone; 3) second ir-
regular reflection zone.
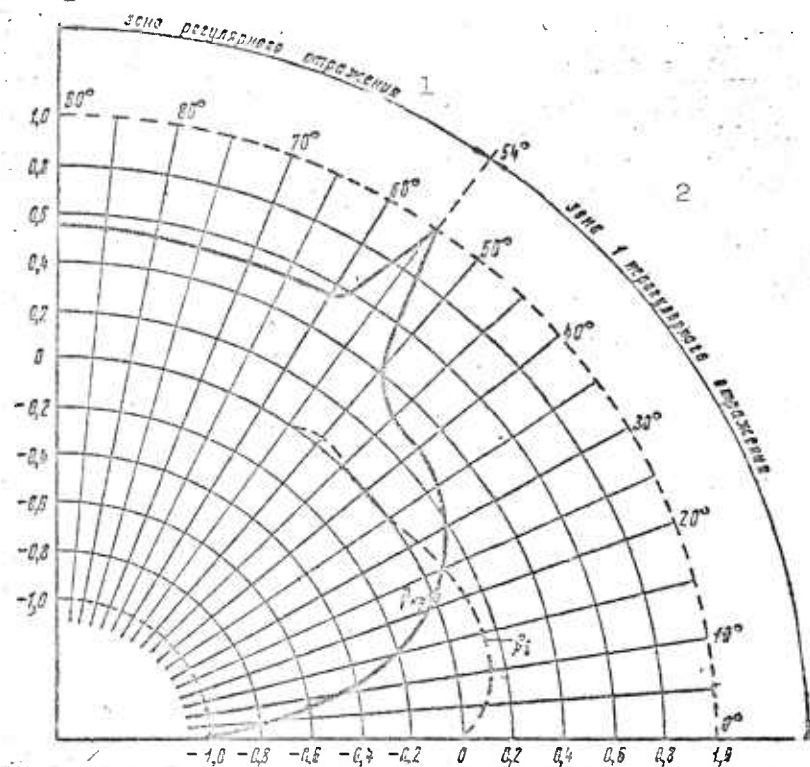
Fig. 68. Angle diagram for clay ground. 1) Re-
gular reflection zone; 2) first irregular re-
flection zone.

layer.

For grounds with velocities $c > a_0 > b$, the wave pattern becomes simpler; there is neither the lateral wave nor the zone of second irregular reflection.

In the case of low-velocity grounds ($c < a_0$) a rarefaction wave is reflected from the bottom of the reservoir for all angles of incidence and attenuates the direct wave in a manner similar to what occurs upon reflection of shock waves from the free surface.

As follows from the angle diagrams, such a phenomenon is observed for all types of grounds, starting with a definite value of the angle of incidence β_{kr}. According to the linear theory, the angle β_{kr} is calculated from the equation

$$[(2-x)^2 - 4\sqrt{1-x}\sqrt{1-\delta_1^2 x}]^2 - \sigma^2 x^4 \frac{\sqrt{1-\delta_1^2 x}}{\delta_2^2 x - 1} = 0, \quad (3.145)$$

where

$$x = \frac{1}{\delta_2^2 \cos^2 \beta_{kp}}. \quad (3.146)$$

In the case when $\beta < \beta_{kr}$, the coefficient A_1 becomes negative and the waves reflected from the bottom of the reservoir become rarefaction waves. This is explained by the fact that in this range of angles of incidence ($0 < \beta < \beta_{kr}$) the front of the direct wave begins to be overtaken by the disturbances caused by the motion of the ground downward from the interface, with velocities greater than the normal components of the ground velocities behind the front of the direct wave.

The deductions of the linear theory of reflection of shock waves from the bottom of a reservoir are qualitatively well confirmed by the experimental data, and can therefore serve as a basis for corresponding approximate estimates.

However, just as in the problem of the influence of the free surface, the waves propagating in either the liquid or in the ground will
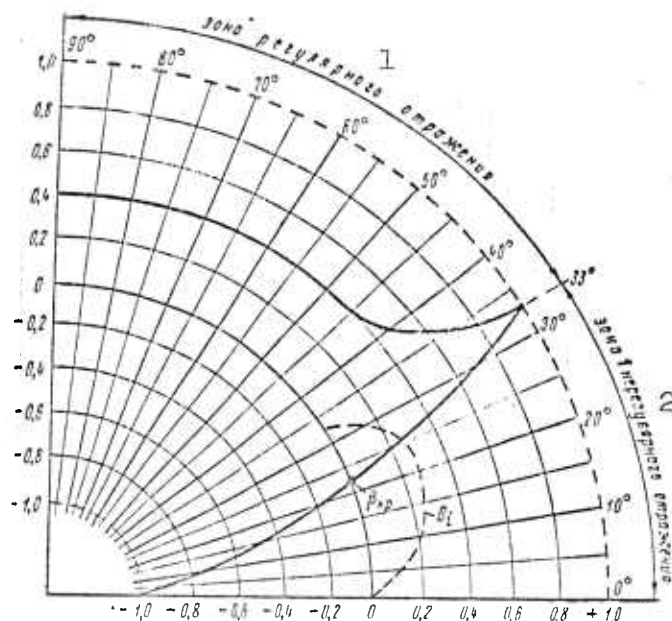
Fig. 69. Angle diagram for sandy ground.
1) Regular reflection zone; 2) first
irregular reflection zone.

have displacement velocities which differ somewhat from their values
in the unperturbed medium.

Actually, as indicated above, a nonlinear superposition of wave
systems takes place. The solution becomes much more complicated. The
exposition of the nonlinear theory is beyond the scope of the present
book. We present below only some general remarks, based on materials
of experimental investigations.

Example. At a distance of 5 m from a rocky bottom (granite, $c =$
$= 5000$ m/sec, $b = 3000$ m/sec, $\rho = 265$ kg-sec^2/m^4) there occurred an
explosion of a TNT charge weighing 1200 kg. Determine the parameters
of the shock wave at the following points of observation:

$$L = 5 \text{ m}, \quad h_1 = 5.0 \text{ m},$$
$$L = 10 \text{ m}, \quad h_1 = 3.0 \text{ m},$$
$$L = 100 \text{ m}, \quad h_1 = 3.0 \text{ m}.$$

Plot the pressure pattern for specified distances and depths.
Neglect the influence of the free surface.

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<u>Solution.</u> To determine β_1, β_2, β_{kr}, A_1, and B_1 we use the available angle diagram for a rocky bottom (granite).

We determine the critical angles with the aid of the angle diagram (Fig. 67): $\beta_1 = 72°30'$, $\beta_2 = 60°$, $\beta_{kr} = 7°$.

We calculate the reflected wave for the first conditions: $L = 5.0$ m, $H_1 = 5.0$ m, $h_1 = 5.0$ m.

The angle of reflection will in this case be

$$\beta = \operatorname{arc\,tg} \frac{H_1 + h_1}{L} = \operatorname{arc\,tg} \frac{5+5}{5} \approx 63°30' < \beta_1.$$

Consequently, the point of observation is in the first zone of irregular reflection. Using the diagram of Fig. 67 (for a rocky bottom), we obtain $A_1 \simeq 79$, $B_1 \simeq 0.045$.

The times of arrival t_{otr} and t_1 are calculated from the formulas (3.136) and (3.141) (we take for δ_1 and δ_2 average values from Table 16):

$$t_{oTP} = \frac{1}{a_0} \sqrt{L^2 + (H_1 + h_1)^2} = \frac{\sqrt{25 + 100}}{1500} = 7.46 \cdot 10^{-3} \operatorname{sec};$$

$$t_1 = \frac{1}{C} \left(L + \frac{H_1 + h_1}{\delta_1} \sqrt{\delta_2^2 - \delta_1^2} \right) =$$

$$= \frac{5 + \frac{10}{0.595} \sqrt{2^2 - 0.595^2}}{5000} = 7.40 \cdot 10^{-3} \operatorname{sec}.$$

The radius of the charge is $R_{03} = 0.053 \sqrt[3]{G} = 0.564$ m, hence

$$\bar{r}_2 = \frac{1}{R_{03}} \sqrt{L^2 + (H_1 + h_1)^2} = \frac{\sqrt{5^2 + 10^2}}{0.564} = 19.8.$$

Finally, we obtain $\theta = 935 \, \bar{r}_2^{0.24} R_{03} \cdot 10^{-6} = 935(19.8)^{0.24} \cdot 0.564 \times$ $\times 10^{-6} = 1.08 \cdot 10^{-3}$ sec. Summing the results in accordance with Formula (3.142), we obtain

$$p_{oTP} = 506 \left[0.79 \sigma_0 (\tau) - 0.045 E_1 \left(\frac{\tau}{\theta} \right) \sigma_0 (t - t_1) \right] e^{-\frac{\tau}{\theta}},$$

where $\tau = t - 7.46 \cdot 10^{-3}$ sec, $\theta = 1.08 \cdot 10^{-3}$ sec.

We calculate p_{otr} for $L = 10$ m, $H_1 = 5.0$ m, $h_1 = 3.0$ m. We determine the angle of reflection

$$\beta \doteq \text{arc tg} \frac{H_1 + h_1}{L} = \text{arc tg} \frac{8}{10} \approx 38°40'.$$

From a comparison of this angle with the critical angles it follows that the point of observation is located in the zone of second irregular reflection. Therefore the calculation of p_{otr} should be carried out by means of Formulas (3.142)-(3.144).

From the angle diagram (Fig. 67) we determine $A_2 \approx 0.905$, $B_2 \approx 0.09$.

For the times t_{otr} and t_1 we obtain

$$t_{отр} = \frac{\sqrt{10^2 + 8^2}}{1500} = 8{,}53 \cdot 10^{-3} \text{ sec},$$

$$t_1 = \frac{10 + \frac{8}{0{,}595} \sqrt{2^2 - 0{,}595^2}}{5000} = 7{,}15 \cdot 10^{-3} \text{ sec}.$$

We have

$$\bar{r}_2 = \frac{\sqrt{10^2 + 8^2}}{0{,}564} = 22{,}8,$$

hence $\theta = 935 [22{,}8]^{0{,}24} 0{,}564 \cdot 10^{-6} = 1{,}12 \cdot 10^{-3}$ sec.

Ultimately we obtain at the given point of observation

$$p_{отр} = 434 \left[0{,}905 \sigma_0(\tau) - 0{,}090 \overline{E}_i \left(\frac{\tau}{\theta} \right) \sigma_0(t - t_1) \right] e^{-\frac{\tau}{\theta}},$$

where

$$\tau = t - 8{,}53 \cdot 10^{-3} \text{ sec};$$
$$\theta = 1{,}12 \cdot 10^{-3} \text{ sec}.$$

The third case is calculated also by means of Formulas (3.142)-(3.144).

After calculations analogous to those given above, we obtain

$$\beta = 4°30', \quad A_2 \approx -0{,}500, \quad B_2 \approx 0{,}015;$$

$$t_{отр} = \frac{\sqrt{100^2 + 8^2}}{1500} = 66{,}7 \cdot 10^{-3} \text{ sec};$$

$$t_1 = \frac{100 + 25{,}7}{5000} = 25{,}1 \cdot 10^{-3} \text{ sec};$$

$$\bar{r}_2 = \frac{\sqrt{100^2 + 8^2}}{0{,}564} = 177;$$

$$\theta = 1{,}83 \cdot 10^{-3} \text{ sec}.$$

Ultimately we obtain for this case
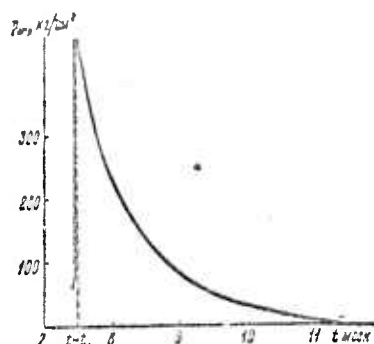
Fig. 70. Pattern of
reflected wave (L =
= 5 m; h_1 = 5 m).
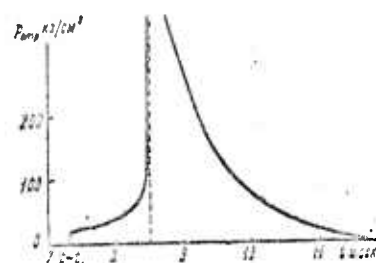
Fig. 71. Pattern of
reflected wave (L =
= 10 m; h_1 = 3 m).
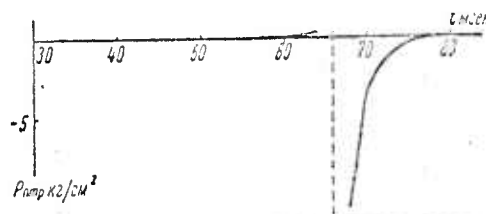
Fig. 72. Pattern of reflected
wave (L = 100 m; h_1 = 3 m).

$$p_{отр} = 42,1 \left[-0,500\sigma_0(\tau) - 0,015E_i\left(\frac{\tau}{\vartheta}\right)\sigma_0(t - t_1) \right] e^{-\frac{\tau}{\vartheta}},$$

where

$$\tau = t - 00,7 \cdot 10^{-3} \text{ sec};$$
$$\vartheta = 1,83 \cdot 10^{-3} \text{ sec}.$$

The pressure vs. time patterns are presented in Figs. 70-72.

§7. QUALITATIVE INFLUENCE OF NONLINEAR EFFECTS UPON REFLECTION OF A SHOCK WAVE FROM THE BOTTOM OF A RESERVOIR

As was mentioned earlier, the linear theory, while correctly describing the mechanism of formation of various wave systems in a liquid, cannot disclose completely the complicated pattern of interaction between the shock wave and the bottom of the reservoir. Two basic causes explain the quantitative discrepancy between the results of the linear theory and the observed phenomena.

1. The assumption that the velocity of propagation of the disturbances in either the liquid or in the ground is constant does not cor-

respond fully to reality. This velocity depends on the amplitude of the compression wave and of the characteristics of the medium, and whereas in water it increases with increasing pressure on the front, in the overwhelming majority of the grounds, to the contrary, waves with larger amplitudes propagate with lower velocities.

2. The grounds can be regarded as linear ideally elastic media only rather approximately. As a rule, the residual deformations of the ground are quite important. As regards such material as silt and sand, it is closer in properties to liquid many-phase media.

In view of this it is necessary to describe the influence of non-linear effects at least qualitatively.

Figures 73 and 74 show typical oscillograms of waves of seismic origin. One can see clearly on them the frontal and lateral waves, the foreshock, and the direct wave. Depending on the distance to the epicenter of the explosion, the frontal and lateral waves are either separated from each other by a definite time interval or, to the contrary, arrive at the point under consideration in the form of some resultant wave disturbance.
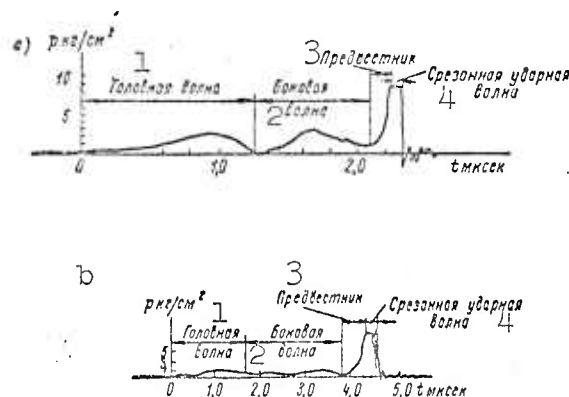
Fig. 73. Oscillograms of waves of seismic origin produced by explosion of charges weighing: a) 20 kg; b) 100 kg. 1) Frontal wave; 2) lateral wave; 3) foreshock; 4) cut-off shock wave.
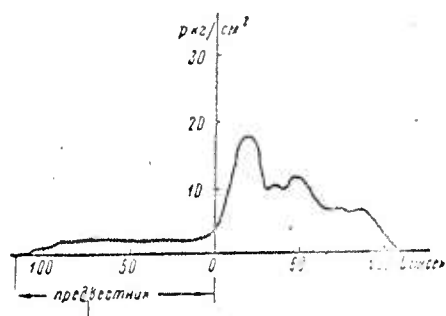
Fig. 74. Character of the pattern of a wave of static origin. The foreshock is clearly seen. 1) Foreshock.

In the case of explosions at considerable distances from the charge to the bottom of the reservoir, the pressure in waves of seismic origin is approximately one order of magnitude lower than the pressure on the front of the shock waves (the amplitude of the direct wave has gone beyond the perforations of the film on the oscillograms presented). As the charge approaches the bottom, the pressure in the frontal and lateral waves increases, while the pressure on the front of the direct wave decreases owing to the reflected pressure-reduction waves.

The intensity of the lateral wave is approximately twice the intensity of the frontal wave.

The maximum amplitude of these waves decreases somewhat more rapidly than $1/L^{3/2}$. Thus, under all conditions, their action at large distances is many times smaller than the action of the direct wave.

The duration of the compression phase of the lateral and frontal waves is usually tens of times larger than the duration of the direct wave. This circumstance makes it possible in many cases to regard the action of the seismic waves on a structure as a static load.

As a rule, at large distances the effect of the lateral and frontal waves can be neglected in practical calculations.

As follows from the oscillograms presented, one observes ahead of the front of the wave a region of smooth growth of pressure — the so-called foreshock. Its mechanism of formation can be represented in simplified form as follows: the direct wave, propagating along the separation boundary, produces wave disturbances in the ground, some of which overtake the front and generate in the liquid an increased-

pressure field. At certain distances there is superimposed on this field the field of the frontal and lateral waves. The maximum pressure in the foreshock amounts to approximately 10-30% of the pressure on the front of the direct shock wave. The lower limit corresponds to explosions at the free surface while the upper limit corresponds to explosions near the bottom of the reservoir. The buildup time of the pressure in the foreshock is comparable with the duration of the positive phase of the pressures in the shock wave.

As was already mentioned earlier, at small angles of incidence of the shock wave, close to the slip angles ($\beta < \beta_{kr}$), the reflected waves are realized by pressure-reduction waves. Propagating with the local velocity of sound, these waves can, under definite conditions, catch up with the front of the direct wave and reduce its amplitude. Such a process is analogous, in its general features, to the process previously considered in the study of the influence of the free surface. The difference consists here in the fact that the rarefaction waves traveling from the free surface are formed at all angles of incidence of the shock waves, and have an intensity equal to the intensity in the direct shock wave, whereas the pressure-reduction waves, propagating from the bottom of the reservoir, are formed at angles of incidence lower than the critical value ($\beta < \beta_{kr}$), and have a lower intensity. Therefore they are unable to attenuate fully the direct wave.

Approximate quantitative estimates of the nonlinear effect of the attenuation of a direct shock wave by rarefaction waves traveling from the bottom of the reservoir can be presented on the basis of the solution of A.A. Grib, A.G. Ryabinin, and S.A. Khristianovich, developed in §3 of the present chapter.

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§8. PARAMETERS OF SEISMIC-EXPLOSION WAVES NEAR THE SURFACE OF THE GROUND

The investigations performed show that the development of the physical processes accompanying an underground explosion is greatly influenced by the presence of the free surface of the ground. When the compression wave reaches the ground surface, the ground begins to move with increased velocity. And for the same reason as in an underwater explosion, a rarefaction wave is produced with amplitudes of the same order of magnitude as the amplitude of the direct wave.

As is well known, rocks and the majority of ground soils can withstand very large compression stresses, but fail readily even under insignificant tension stresses. Consequently, the region of the ground located in the direct vicinity of the free surface of the ground, becomes a region of main damage. A funnel of conical form is produced, the dimensions of which are usually determined by its radius R_v and depth H_v.

The radius of the funnel can be determined from Boreskov's empirical formula

$$R_u = mH^3 (0{,}4 + 0{,}6n^3), \qquad (3.147)$$

where n is the so-called Horn exponent, frequently also called the explosive action exponent; H is the depth of the charge; and m is a coefficient characterizing the type of the ground.

$$n = R_v/H. \qquad (3.148)$$

According to G.I. Pokrovskiy

$$m = 0{,}80 + 0{,}085N, \qquad (3.149)$$

where N is the strength category of the rock (N varies from 1 to 16 on the 1944 scale of unified norms for production and wages).

For TNT the Boreskov formula can be written in dimensionless coordinates in the form

$$\bar{R}_{\text{в}} = 1{,}2\bar{H} \left[\frac{6{,}75 \cdot 10^3}{m\bar{H}^3} - 0{,}4 \right]^{1/3}.$$ (3.150)

Formula (3.150) gives good accuracy for the range

$$1 < n = \frac{R_\text{в}}{H} < 2.$$

The depth of the funnel is estimated with the aid of the empirical relation

$$\bar{H}_\text{в} = \frac{2\bar{R}_\text{в} - \bar{H}}{3}.$$ (3.151)

On the basis of theoretical considerations and of an analysis of modern experimental data, G.I. Pokrovskiy proposed for the calculation of the funnel radius an equation that admits of wider extrapolation than Boreskov's formula. This equation has the form

$$R_\text{в} = \frac{\gamma}{28\,000} H^{\frac{7}{2}} (1 + n^2)^2,$$ (3.152)

where γ is the specific gravity of the ground in kg/m^3.

The volume of ground expelled under optimum depth of the charges in the ground reaches one or one and a half cubic meters of rock per kilogram of explosive.

As was already mentioned earlier, the values of the normal stresses in soft grounds, for the middle zone of underground explosion, is calculated from the formula

$$\sigma_{r\max} = \frac{425Fk}{R^3} \ \text{kg/cm}^2.$$ (2.165)

The coefficient F depends on the depth of the charge. Its values are listed in Table 17.

In the case when the depth of the charge and the depth of the measurement point do not exceed 10 radii of the charge we have

$$\sigma_{r\max} = \frac{3200Fk}{R^4}.$$ (3.153)

In Formulas (2.165) and (3.153) the coefficient $\underline{k}$ is taken in accordance with the data of Table 13.

TABLE 17

1 Глубина заложения заряда H	0	3	6	9	12	15
2 Значение F	0,20	0,46	0,67	0,86	0,96	1,0

1) Depth of charge H; 2) value of F.

For solid rock of the type of granite, limestone, and marble, the value of the maximum stress is determined by the empirical relation (2.166)

$$\sigma_{r\max} = \frac{10^{v}F}{\rho a}\left[\frac{95,1}{R} - \frac{70,1 \cdot 10^{2}}{R^{2}} + \frac{26,6 \cdot 10^{4}}{R^{3}}\right] \text{ kg/cm}^2.$$

In the same way as in an unbounded medium, the stress and the particle velocity are linearly related:

$$v_{r\max} \simeq \frac{\sigma_{r\max}}{\rho a} \cdot 10^{5} \text{ cm/sec.} \tag{3.154}$$

According to American data* the acceleration of the ground particles at the free surface can be calculated from the formula

$$A_{g} = 6,3\frac{Fk}{\rho R_{03}}\left[\frac{38,4 \cdot 10^{4}}{R^{4}} + \frac{0,17 \cdot 10^{2}}{R^{2}} + \frac{0,31}{R}\right], \tag{3.155}$$

where A_g is the acceleration as a fraction of the acceleration due to gravity; R_{03} is the radius of the charge in meters; ρ is the density of the ground, kg-sec^2/m^4.

In the far zone of underground explosions ($100 < \overline{R} < 1000$) the main interest attaches to the velocity of the ground at the surface. The maximum values of the velocity are calculated from the formula of M.A. Sadovskiy

$$v_{\max} = \frac{408}{\sqrt[3]{f(n)}}\,\overline{R}^{-\frac{1}{2}}\,\text{cm/sec}, \tag{3.156}$$

where f(n) is Horn's exponential function; γ is the specific gravity of the explosive in kg/cm^3; R is the relative distance.

Horn's exponent is approximated by the following approximate rela-

tions:

$$\sqrt[3]{f(n)} = \begin{cases} 1{,}7 & \text{for} \quad n > 2 \\ 0{,}4 + 0{,}6n^3 & \text{for} \quad 2 \geqslant n \geqslant 1 \\ 1 & \text{for} \quad n < 1 \end{cases} \tag{3.157}$$

For TNT ($\gamma = 1600$ kg/m^3) Formula (3.156) assumes the form

$$v_{max} = \frac{16\,300}{\sqrt[3]{f(n)} \cdot R^{1,5}} \quad \text{cm/sec.} \tag{3.158}$$

As a result of the propagation of the wave disturbances, the ground in the far zone executes a series of complicated vibrations. Usually one separates, from this series, the principal phase characterized by the largest ground displacements. In the principal phase of the vibrations, these displacements along the surface can be determined approximately by the relationship

$$u(t) = A_0 e^{-\alpha t} \sin \omega t \quad \text{mm,} \tag{3.159}$$

where A_0 is the maximum amplitude of the ground-particle displacement; $\omega = 2\pi/T$ is the circular frequency of the vibrations; α is the damping coefficient.

The value of α depends on the period of the vibrations and is determined from the data of Table 18.

TABLE 18

T, сек.	0,111	0,083	0,067
α, сек.$^{-1}$	0,020—0,025	0,036—0,040	0,055—0,060

1) T, sec; 2) α, sec^{-1}.

The period of the vibrations in the principal phase is calculated from the formula of M.A. Sadovskiy

$$T = a_2 \lg R \quad \text{sec,} \tag{3.160}$$

where α_2 is a coefficient that depends on the physical and mechanical properties of the ground and R is the distance from the center of the explosion in meters.

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The values of the coefficient α_2 for some types of ground are listed in Table 19.

TABLE 19

1 Вид грунта	2 Коэффициент α_2
3 Грунты водонасыщенные	0,13—0,15
4 Скальные породы	0,03—0,04
5 Глина, лёсс	0,09—0,095

1) Type of ground; 2) coefficient; 3) water-saturated ground; 4) solid rock; 5) clay, loess.

The maximum amplitude of ground particle displacement is determined from the formula

$$A_0 = 205 \frac{T}{2\pi} \gamma^{1/3} \bar{R}^{-1} \text{ mm.} \qquad (3.161)$$

For TNT

$$A_0 = 240 \frac{T}{2\pi} \frac{1}{\bar{R}} \text{ mm.} \qquad (3.162)$$

Example. Calculate the velocities of the ground at the free surface at a distance of 500 m from the center of explosion of a TNT charge weighing 50 tons in firm ground (saturated clay, N = 12). Find the value of the overload at the same distance.

The depth of the charge is assumed to be 5 m.

Solution. We determine the radius of the equivalent spherical charge R_{03}, the relative distance of the investigated point from the center of the explosion, and the relative depth of location of the charge:

$$R_{03} = 0,053 \sqrt[3]{50 \cdot 10^3} = 0,053 \cdot 10 \cdot 3,68 = 1,95 \text{ m};$$

$$\bar{R} = \frac{500}{1,95} = 256; \quad \bar{H} = \frac{5}{1,95} = 2,56.$$

To estimate the maximum velocity we use the formula of M.A. Sadov-

skiy

$$v_{max} = \frac{16\,300}{\sqrt[3]{f(n)} \cdot \overline{R}^{1,6}} \text{ cm/sec.} \qquad (3.156)$$

For this purpose it is first necessary to calculate Horn's exponent $n = R_v/H$. According to Boreskov's formula

$$n = \frac{R_n}{H} = 1{,}2 \left[\frac{6{,}75 \cdot 10^3}{mH^3} - 0{,}4 \right]^{1/3}. \qquad (3.150)$$

Substituting in it

$$m = 0{,}80 + 0{,}085N = 0{,}80 + 0{,}085 \cdot 12 = 1{,}82 \qquad (3.149)$$

and the indicated value of $\overline{R}$, we obtain

$$n = 1{,}2 \left[\frac{6{,}75 \cdot 10^3}{1{,}82 \cdot 2{,}56^3} - 0{,}4 \right]^{1/3} \approx 7{,}2.$$

For $n > 2$ we have

$$\sqrt[3]{f(n)} = 1{,}7,$$

and therefore

$$v_{max} = \frac{16\,300}{\sqrt[3]{f(n)}\ \overline{R}^{1,6}} = \frac{16\,300}{1{,}7 \cdot 4100} = 2{,}34 \text{ cm/sec.}$$

We determine the value of the overload by means of Formula (3.155). The values of F, k, and ρ are found from Tables 13 and 17: F = 0.393; k = 7000 kg/cm^2; ρ = 215 kg-sec^2/m^4.

Substituting the initial data in (3.155), we obtain

$$A_g = 6{,}3 \frac{Fk}{\rho R_{03}} \left[\frac{38{,}4 \cdot 10^4}{\overline{R}^4} + \frac{0{,}17 \cdot 10^2}{\overline{R}^2} + \frac{0{,}31}{\overline{R}} \right] =$$

$$= \frac{6{,}3 \cdot 0{,}393 \cdot 7000}{215 \cdot 1{,}95} \left[\frac{38{,}4 \cdot 10^4}{256^4} + \frac{0{,}17 \cdot 10^2}{256^2} + \frac{0{,}31}{256} \right] =$$

$$= 41{,}2 \left[0{,}895 \cdot 10^{-4} + 0{,}026 \cdot 10^{-2} + 0{,}121 \cdot 10^{-2} \right] = 41{,}2 \cdot 0{,}156 \cdot 10^{-2} = 0{,}0643.$$

[Footnotes]

182   See, for example, L.I. Briggs, J. Appl. Phys., 21, 721, 1950.

189   A.A. Grib, A.G. Ryabinin, S.A. Khristianovich, On the Reflec-
tion of a Plane Shock Wave in Water from the Free Surface,
Applied Mechanics and Mathematics, Vol. XX, No. 4, AN SSSR,
1956.

192   When $n = 7.15$, the discrepancy amounts to less than 0.1% for
pressures near 1000 atm.

196   Here and throughout the exposition of the present section we
follow the terminology of S.A. Khristianovich.

198   In the x,z plane the front of the pressure-reduction wave is
a circle of radius at with centerline on the free surface of
the liquid.

199   The second root of the equation $Z'_v = \beta^* + \beta_{pr}$, which lies
in the region of positive Z (in the upper half plane) is dis-
carded.

200   See, for example, E. Kamke, Handbook of Ordinary Differen-
tial Equations, IIL, 1950.

209   See, for example, Ya.I. Frenkel', Kinetic Theory of Liquids,
AN SSSR, 1945; Ya.B. Zel'dovich, J. Exptl. Theoret. Phys.,
12, 525, 1942.

214   The first term in the brackets is a consequence of the as-
sumption that the pressure is zero in the region of the cavi-
tational discontinuity.

220   This range of acoustic ground impedances corresponds to a
value of the reflection coefficient ranging from −0.8 to
+0.7 for direct incidence.

221   The Lame constant μ is equal to the shear modulus G. The con-
nection between the constant λ, the modulus of longitudinal
elasticity E, and the Poisson coefficient ν is expressed by
the relation

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}.$$

224   As already mentioned earlier, the connection between the Lame
constants and the velocity of propagation of the longitudinal
and transverse waves in a medium is expressed by the rela-
tionships

$$c = \sqrt{\frac{\lambda + 2\mu}{\rho}}, \qquad b = \sqrt{\frac{\mu}{\rho}}.$$

[Footnotes (Continued)]

225   *Ye.I. Shemyakin, K.I. Markova, Propagation of Nonstationary
Disturbances in a Liquid in Contact with an Elastic Half
Space, Applied Mathematics and Mechanics, Vol. XXI, No. 1,
1957.

225   **Here and throughout, foregoing a rigorous exposition, we as-
sume the amplitude of the wave to vary with distance not as
$1/r$, as would follow from the equations of acoustics, but as
$1/r^{1.13}$, as would follow from the empirical formula (2.150).

227   * $E_i(t) = \int_{-\infty}^{x} \frac{e^t}{t} dt.$ See, for example, Yanke and Emde, Tables of Func-

tions with Formulas and Curves, OGIZ, GITL, 1948.

240   "The Effects of Atomic Weapons," New York-Toronto-London,
1950.

[List of Transliterated Symbols]

174   отр = otr = otrazhennaya volna = reflected wave

174   ф = f = front = front

175   пред = pred = predel'noye otrazheniye = limit reflection

175   з = z = zadannyy = given

175   опт = opt = optimal'nyy = optimum

183   ак = ak = akusticheskoye priblizheniye = acoustic approxi-

mation

184   мксек = μsec

190   хар = khar = kharakteristiki = characteristics

191   в.р = v.r = volna razrezheniya = rarefection wave

193   пр = pr = pryamaya volna = direct wave

207   кг/см^2 = kg/cm^2

210   к = k = kavitatsiya = cavitation

210   рез = rez = rezul'tiruyushcheye davleniye = resultant pressure

213   атм = atm = atmosfernoye protivodavleniye = atmospheric
              counterpressure

215   соуд = soud = soudareniye = collision

222   д = d = dno = bottom

226   г = g = grunt = ground

230   кр = kr = kriticheskiy = critical

234   мсек = msek = millisekunda = millisecond

235   мксек = μsec

238   в = v = voronka = funnel

Chapter 4

PRINCIPAL ASPECTS OF THE PROBLEM OF EXTERNAL FORCES
IN THE CASE OF AERIAL AND UNDERWATER EXPLOSIONS

§1. CONCEPTION OF DIFFRACTION PROBLEMS

The very simple boundary-value problems of explosion theory considered in the preceding chapter presupposed that the surfaces of separation between the two media were unbounded. Such an assumption, naturally, cannot be regarded as justified when a shock wave encounters a partition of finite dimensions. In this case there occurs along with reflection and refraction of the waves also a process wherein the shock wave bends around the partition, usually called diffraction.

The study of diffraction phenomena is of very great practical significance, for without it it is impossible to present an estimate of the external forces in an explosion.

Very useful for a qualitative understanding of the diffraction process is the Huygens principle, which consists, as is well known, in the fact that each point of the wave or of the surface with which the wave interacts can be regarded as an elementary source of wave disturbances. Using this concept, it is easy to construct the front of a propagating wave (Fig. 75) and to indicate the main qualitative features of diffraction phenomena. Thus, for example, if a plane wave encounters a

Fig. 75. Construction of wave fronts in an accordance with the Huygens principle.

screen provided with a small aperture, then this aperture will serve as the source of an outgoing wave on the other side of the screen.

A mathematical analysis of the diffraction problem was first presented in a finished form by Kirchhoff. For a homogeneous wave equation in three-dimensional space, he obtained a rather general relationship, which determines the potential at any point of space in terms of the values of this potential and its derivatives on an arbitrary closed surface, enclosing the region in which the chosen point is situated.

Kirchhoff's formula has the form

$$\varphi(\xi, \eta, \zeta, t) = \frac{1}{4\pi} \int_s \left\{ -\varphi\left(x, y, z, t - \frac{r}{a_0}\right) \frac{\partial \frac{1}{r}}{\partial n} + \right.$$

$$\left. + \frac{1}{r}\left(\frac{\partial\varphi\left(x, y, z, t - \frac{r}{a_0}\right)}{\partial n}\right) + \frac{1}{a_0 r}\frac{\partial r}{\partial n}\frac{\partial\varphi\left(x, y, z, t - \frac{r}{a_0}\right)}{\partial t}\right\} ds, \qquad (4.1)$$

where $\varphi(\xi,\ \eta,\ \zeta,\ t)$ is the potential at the point $A(\xi,\ \eta,\ \zeta)$, enclosed by the arbitrary surface $\underline{s}$; $\overrightarrow{n}$ is the outward normal to the surface; $\underline{r}$ is the distance from the element $\underline{ds}$ to the point $A(\xi,\ \eta,\ \zeta)$.

In spite of the completeness of the mathematical solution, analysis of diffraction problems with the aid of the Kirchhoff formula encounters difficulties of principal character.

The point is that although the surface on which the values of the potential and its normal derivative must be known can be chosen arbitrarily, in most physical problems there are no reliable data concerning these quantities on the chosen surface until the problem is solved. It therefore becomes necessary to make additional assumptions concerning the distribution of the potential and its derivatives on the boundary surface, which in the best case leads to a method of successive approximations.

It must be added to it that the surface integrals contained in

(4.1) cannot be evaluated in close form even for the simplest cases. Consequently, one uses frequently for an approximate analysis of diffraction problems not the Kirchhoff formula, but the so-called radiation integral, which represents a mathematical form for writing down the Huygens principle. This integral has the form

$$\varphi_A = \frac{1}{2\pi} \int_s \frac{v_n\left(t - \frac{r}{a_0}\right)}{r}\, ds, \qquad (4.2)$$

where v_n is the normal component of the velocity of the surface element $\underline{ds}$ and $\underline{r}$ is the distance from $\underline{ds}$ to the point A.

Its use is possible only when the point at which the potential is estimated is on the "line of sight" with respect to the arbitrary surface element.

Let us indicate the sequence of operations connected with the application of the radiation integral in diffraction problems. Assume, for example, that it is necessary to find the pressure field in a liquid, with allowance for the influence of an absolutely rigid plane partition of specified configuration. In this case one first calculates the normal components of the particle velocities at those points of the liquid, which are in direct contact with the surface of the partition, under the assumption that the partition itself is missing. The obtained quantities with their signs reversed are substituted in the radiation integral.

The sum of the potential of the direct wave and the potential obtained by this method determines the sought pressure field. It is obvious here that the following boundary condition is satisfied on the surface of the partition:

$$\frac{\partial \varphi}{\partial n}\bigg|_s = 0.$$

The foregoing methods, which follow from the general theory of

the wave potential, are valid only for an analysis of linear equations.

The propagation of aerial shock waves cannot as a rule be described by linear equations. Consequently, there is likewise no rigorous theory of the diffraction of such waves. For practical estimates one uses the approximate representations formulated in the papers of Yu.B. Khariton, M.A. Sadovskiy, and many other authors.

Diffraction problems pertaining to an underwater explosion can likewise not be solved in many cases with sufficient accuracy on the basis of the linear equations indicated here.

§2. DIFFRACTION OF AN AERIAL SHOCK WAVE AROUND A PARTITION

The problem of the diffraction of an aerial shock wave around a partition will be considered under the following assumptions:

1) a plane wave of constant amplitude is investigated;

2) the wavelength is infinite.

When such a wave is normally incident on an absolutely rigid wall, a reflection pressure is produced, which can be determined from the Crussard-Izmaylov formula

$$\Delta p_{\text{отр}} = 2\Delta p_{\Phi} + \frac{6\Delta p_{\Phi}^2}{\Delta p_{\Phi} + 7p_0}.$$ (1.221)

The reflection pressure remains at the specified point of the partition until the instant when the rarefaction wave, moving from the edges to the center, arrives at this point.

The pressure will then be determined approximately by the parameters of the stagnant liquid stream. The values of such parameters can be readily obtained from the Bernoulli integral, according to which

$$\frac{a^2}{k-1} + \frac{v^2}{2} = \frac{a^*}{k-1},$$ (1.84)

where a* denotes the value of the velocity of sound at the point where the gas velocity is equal to zero (in this case, on the wall).

Recognizing that $a^{*2}/a^2 = p^*\rho/p\rho^*$, we can represent Eq. (1.84)

in the following fashion:

$$\frac{p^*}{p}\frac{\rho}{\rho^*} = 1 + \frac{k-1}{2}\frac{v^2}{a^2}. \tag{4.3}$$

Here p* and ρ* denote the pressure and density in the retarded stream.

Using the adiabatic condition p/ρ^k = const, it is easy to eliminate the parameter ρ from Relation (4.3). As a result we obtain

$$\left(\frac{p^*}{p}\right)^{\frac{k-1}{k}} = 1 + \frac{k-1}{2}\frac{v^2}{a^2},$$

or

$$p^* = p\left\{1 + \frac{k-1}{2}\frac{v^2}{a^2}\right\}^{\frac{k}{k-1}}. \tag{4.4}$$

The quantities v and a for an infinitely long wave can be assumed on the basis of the equations of dynamic compatibility:

$$a = a_0\sqrt{\frac{p}{p_0}\frac{\frac{k+1}{k-1} + \frac{p}{p_0}}{\frac{k+1}{k-1}\frac{p}{p_0} + 1}}; \tag{1.187}$$

$$v = a_0\frac{\left(\frac{p}{p_0} - 1\right)}{k\sqrt{\frac{k+1}{2k}\frac{p}{p_0} + \frac{k-1}{2k}}}. \tag{1.189}$$

After elementary transformations we obtain ultimately

$$p^* = p\left\{1 + \frac{(p-p_0)^2}{kp\left(\frac{k+1}{k-1}p_0 + p\right)}\right\}^{\frac{k}{k-1}}. \tag{4.5}$$

The last relation was established in a somewhat different manner by Yu.B. Khariton.

After the complete termination of the diffraction processes and the onset of the steady-state flow mode, the force acting on the partition will be determined by the relation

$$F = \bar{c}_F S\frac{\rho v^2}{2}, \tag{4.6}$$

where $\bar{c}_F$ is the coefficient of aerodynamic force and S is the characteristic area of the surface of the partition.

$$\frac{\rho v^2}{2} = 2.5 \frac{(p - p_0)^2}{6p_0 + p}. \tag{1.192}$$

Resolving this total aerodynamic force $\mathbb{F}$ along the axes of the velocity system of coordinates, we obtain

$$\begin{cases} X = c_x \frac{\rho}{2} S v^2, \\ Y = c_y \frac{\rho}{2} S v^2, \\ Z = c_z \frac{\rho}{2} S v^2. \end{cases} \tag{4.7}$$

Here c_x is the coefficient of the resistance force, c_y is the co-efficient of the lifting force; and c_z is the coefficient of the lateral force.

The values of these coefficients are generally found experimentally in wind-tunnel tests.

Figure 76 shows a comparison of the reflection pressure Δp_{otr}, the pressure in the stagnant flow Δp^*, and the pressure in the stationary flow $\Delta p_{rez} = c_x \rho v^2/2$ for a rectangular plate located along the normal to the flow direction. We see that when $\Delta p < 3.5$ atm* the ratio $\Delta p^*/\Delta p_{otr}$ is approximately equal to one half.

This fact enables us to outline the following approximate scheme for estimating the external forces produced on the partition in the case of an aerial explosion.

1. The reflection pressure is approximated by a triangular pattern (Fig. 77). The abscissas represent the time of arrival of the rarefaction wave from the edge of the partition at the specified point, $t_0 = l/a$.

2. The ordinate corresponding to the abscissa t_0 is bisected. The resultant point is connected with the point on the abscissa axis corresponding to the time of the positive pressure phase.

If the problem consists of constructing the function $\Delta p = f(t)$ not for one point but for the entire partition as a whole, then the

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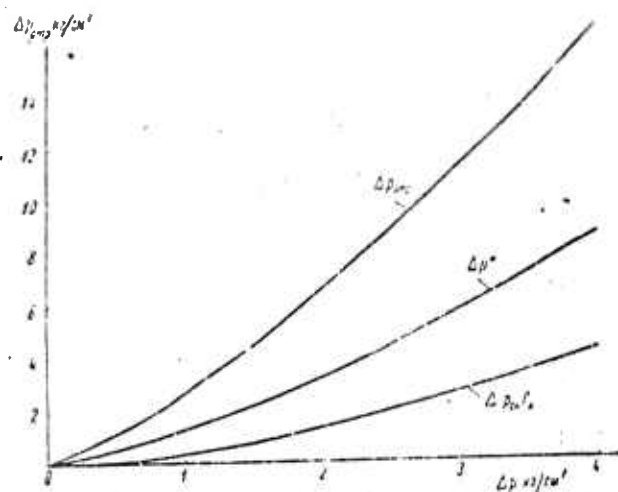
Fig. 76. Reflection pressure and
velocity head as compared with the
pressure in the stagnant flow.
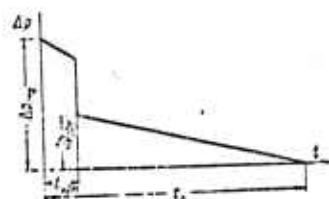
Fig. 77. Pressure
pattern at a speci-
fied point of a par-
tition with finite
dimensions.
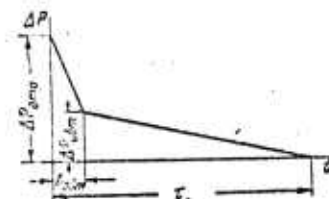
Fig. 78. Resultant
load on the face
side of a partition
of finite dimensions.
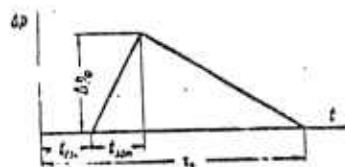
Fig. 79. Resultant
load on the shadow
side of the parti-
tion.
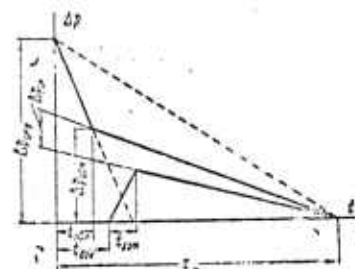
Fig. 80. Resultant
load on a partition
of finite dimensions.

average pressure is determined by integrating the pressures at points
with various distances from the edges. In this case, owing to the fact
that the pressure decreases rather rapidly at the points that are

closest to the edges, the curve assumes the form shown in Fig. 78.

On the first section $0 < t < t_{obt}$, the pressure pattern is approximated by the linear relationship

$$\Delta p = \Delta p_{\text{отр}}\left(1 - \frac{t_{\text{обт}}}{t_1}\right), \qquad (4.8)$$

where $t_1 = 2t_{obt}$, t_{obt} is the travel time of a sound wave (a $\simeq$ 360-400 m/sec) along the smaller of the dimensions of the structure.

On the second section it is likewise perfectly permissible to use the linear approximation

$$\Delta p = \Delta p^{*}\left(1 - \frac{t_{\text{обт}}}{t_2}\right), \qquad (4.9)$$

where $t_2 = \tau_+ - t_{obt}$.

The pressure on the surface facing the side opposite the explosion center can be approximately regarded as equal to the maximum pressure on the wave front for a load buildup time $t_{zat} \simeq l/a_0$ and for a linear decrease in the amplitude (Fig. 79).

Thus, the resultant load on the partition is characterized by the pattern shown in Fig. 80.

In spite of the rather approximate character of the developed scheme, it is presently accepted as the main scheme for the estimate of external forces acting on a structure during an aerial explosion.

§3. PROPAGATION OF AN AERIAL SHOCK WAVE IN CHANNELS

Among the important problems involved in the estimate of external forces occurring in aerial explosions is the study of the influx and propagation of shock waves in channels of various types.

Rigorous solution of this problem entails great difficulties. At the present time one can only indicate, making use of the theory of a point explosion, a limiting law for the attenuation of the amplitude of the plane wave.

In accordance with the solution of L.I. Sedov we have

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$$N = \frac{2}{2+\nu} \frac{r_\phi}{t_\phi}, \qquad (2.41)$$

with

$$\lambda^* = \frac{E_0}{\rho_0} \frac{t^2}{r^{2+\nu}}. \qquad (2.35)$$

Putting $\nu = 1$, we obtain

$$N = \frac{2}{3} \sqrt{\frac{E}{\lambda^* \rho_0}} \frac{1}{\sqrt{r_\phi}}, \qquad (4.10)$$

and since

$$p_\phi \simeq \frac{2}{k+1} \rho_0 N^2, \qquad (2.39)$$

we have

$$p_\phi \sim \frac{1}{r_\phi}. \qquad (4.11)$$

The experimental study of the propagation of aerial shock waves in channels of constant and variable cross section has engaged many workers. The latest and fullest results belong to M.A. Sadovskiy, Yu.N. Ryabinin, V.N. Rodionov, and Yu.S. Vakhromeyev.

We present below the final relations necessary for practical calculations.

It has been established that the variation of the pressure on the wave front as the wave propagates in a channel of constant cross section is characterized, in full agreement with (4.11), by the relationship

$$\Delta p_x = \frac{\Delta p_1 l_1}{l_1 + x}, \qquad (4.12)$$

where Δp_1 is the excess pressure in the wave, measured at a certain section of the tube and chosen as the initial value; l_1 is the distance from the center of the charge corresponding to Δp_1; and $\underline{x}$ is the variable distance from the section l_1.

Frequently one assumes for the initial values of Δp_1 and l_1 the excess pressure Δp_0 of the wave at the inlet and the distance r_0 from the center of the charge to the inlet to the channel.

In view of the fact that the formation of a plane shock wave oc-
curs in the channel after the entrance into the channel of a wave of
spherical form and that such a process depends essentially on the
value of the pressure on the front, a correction to Formula (4.12) is
introduced. One chooses for l_1 not the distance from the center of the
charge to the inlet to the channel, but some smaller quantity l_0,
equal to

$$l_0 = \alpha r_0.$$
$$(4.13)$$

The values of the experimentally obtained coefficient α are shown
in Fig. 81.
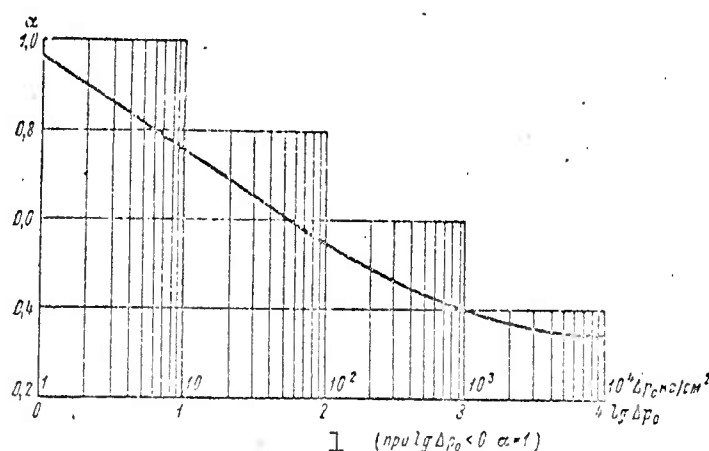
Fig. 81. Dependence of the coefficient
α on the pressure on the front of a
shock wave at the inlet section of the
channel. 1) for log $\Delta p_0 < 0$, $\alpha=1$.

The value of Δp_0 is calculated by the formula (2.145) of M.A.
Sadovskiy with $r = r_0$. Usually the inlet to the channel has a reflect-
ing surface. The presence of such a surface increases the pressure in
the inflowing wave.

Yu.N. Ryabinin and V.N. Rodionov presented an approximate esti-
mate of the coefficient Ω, taking into account the influence of the
reflecting surface, with

$$\Omega = \Delta p_0/\Delta p_1,$$
$$(4.14)$$

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where Δp_0 is the pressure before the entrance into the channel and Δp_1 is the pressure at the beginning of the channel which has a reflecting surface.

They based their calculations on the following assumptions:

the wavelength, and consequently also the thickness of the compressed-air layer, is much larger than the diameter of the inlet channel;

the reflecting surface has infinite dimensions.

It is then possible to indicate four zones corresponding to definite states of the gas at the initial period of the inflow of the wave (Fig. 82):

1) zone "a" — undisturbed air;

2) zone "b" — air in the channel through which a shock wave has passed;

3) zone "c" — air flowing into the channel;

4) zone "d" — air compressed by the reflected wave.

The transition of the gas from state "a" into state "b," is through a nonstationary strong discontinuity surface (shock wave front). Zone "b" is separated from zone "c" by a contact discontinuity. In zones "a" and "d" the pressure p, the density ρ, and the velocity of the particles v are known.

Fig. 82. Pattern of inflow of a shock wave into a channel of constant cross section.

The problem reduces to determining the six quantities p_b, ρ_b, v_b, p_c, ρ_c, and v_c.

To estimate these quantities we can use the following equations:
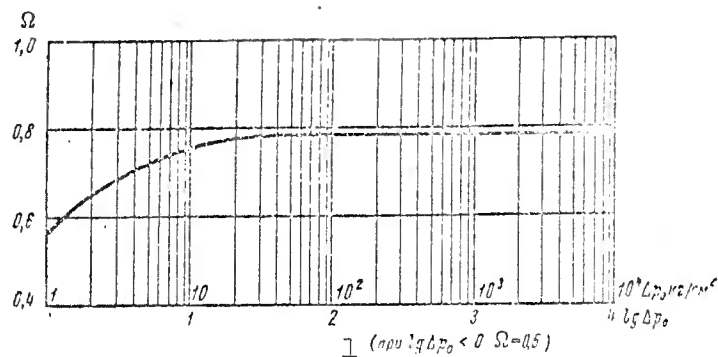
Fig. 83. Dependence of the coefficient Ω on the pressure on the shock-wave front at the inlet section. 1) for $\log \Delta p_0 < 0$, $\Omega = 0.5$.

$$p_6 = p_\text{u} \\ v_6 = v_\text{n}$$

from the condition on the contact surface;

$$\frac{p_6}{p_a} = \frac{p_6 (k+1) + p_a (k-1)}{p_6 (k-1) + p_a (k+1)}, \\ v_6 = \sqrt{(p_6 - p_a)\left(\frac{1}{p_a} - \frac{1}{p_6}\right)}$$

from the dynamic compatibility conditions on the front of the shock wave;

$$v_\text{n} = \sqrt{\frac{2k}{k-1} \frac{p_\text{r}}{p_\text{r}} - \frac{p_\text{n}}{p_\text{u}}}$$

on the basis of the Bernoulli integral;

$$\frac{p_\text{r}}{p_\text{u}} = \left(\frac{p_\text{r}}{p_a}\right)^k$$

from the adiabaticity of the process.

From the solution of this system we can obtain the value of Δp_b, and consequently also the coefficient $\Omega = \Delta p_0 / \Delta p_b$.

The results of the corresponding calculations are shown in Fig. 83.

In addition to the foregoing factors, the attenuation of the shock wave in the channel is influenced also by losses to friction connected with the roughness of the walls.

Generalization of the experimental data enables us to arrive at the following conclusions:

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1) the influence of the wall roughness is determined approximately by the relation

$$\frac{\Delta p_{\text{кан}}}{\Delta p} = e^{-k\frac{x}{d}},\qquad (4.15)$$

where k is a coefficient that depends on the relative roughness; d is the channel diameter; and x is the path covered by the wave along the channel;

2) the ratio of the coefficient k to the hydraulic coefficient of friction ζ is approximately constant and equal to 0.4.*

The character of the dependence of the hydraulic coefficient of friction on the relative roughness of the channel wall and on the Reynolds number is shown in Fig. 84. In the self-similar region one can recommend the following approximating formula:

$$\zeta = \frac{1}{\left(2\lg\frac{d}{2k} + 1.74\right)^2},\qquad (4.17)$$

where d/2k is a quantity inverse to the relative roughness; d is the diameter of the channel; k is the height of the roughness projection. Gathering together the estimates obtained, we arrive at the following final relationship for the pressure in the propagation of a shock wave in a channel of constant cross section:

$$\Delta p = \frac{\Delta p_1\,(l_0 + l_1)}{l_0 + l_1 + x}\,e^{-0.4\zeta\frac{x}{d}}\qquad (4.18)$$

or, in the particular case when $l_1 = 0$ (the inlet section of the channel is regarded as the starting point)

$$\Delta p = \frac{\Delta p_0 \alpha r_0}{\Omega\,(\alpha r_0 + x)}e^{-0.4\zeta\frac{x}{d}}.\qquad (4.19)$$

The value of Δp_0 is calculated by Formula (2.145) for a distance equal to r_0. The coefficients α and Ω are picked off the curves on Figs. 81 and 83.

For an appreciable attenuation of the shock wave in a channel of
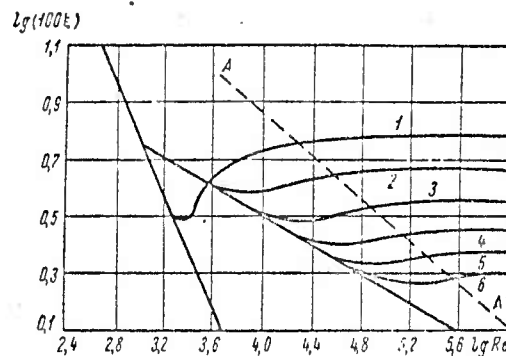
Fig. 84. Dependence of the coefficient ξ on the Reynolds number and on the relative roughness

$$1 - \frac{a}{2k} = 15,0$$

$$2 - \frac{a}{2k} = 30,6$$

$$3 - \frac{a}{2k} = 60,0$$

$$4 - \frac{a}{2k} = 126$$

$$5 - \frac{a}{2k} = 252$$

$$6 - \frac{a}{2k} = 507$$
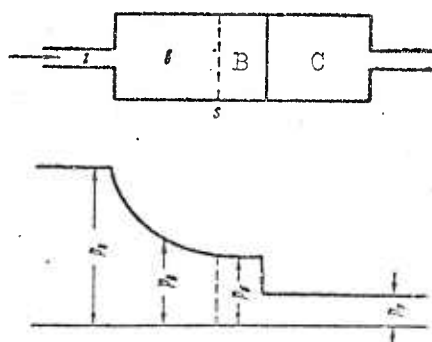
Fig. 85. Scheme of inflow of a shock wave into an expansion chamber.

short length, various types of protective devices are used. One of the most widely used protective devices is an expansion chamber.

The wave-suppression ability of the chamber is estimated by means of the extinction coefficient γ, which is equal to the ratio of the pressure of the shock wave in a channel of constant cross section to the pressure behind the expander, taken in both cases at an equal distance from the charge and for identical cross sections of the linear supply portions of the channel.

It has been experimentally established that the extinction coefficient is not connected directly with the volume, but depends strongly on the diameter of the chamber, increasing with the latter, and varies

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as a function of the intensity of the shock wave entering into the expander.

An approximate method of calculating the coefficient of extinction of an expansion chamber was developed by Yu. N. Ryabinin and V. N. Rodionov. The gist of the method is as follows.

Consider a wave of infinite length and of high intensity. As before, we investigate four regions of the state of the gas (Fig. 85).

On the basis of the mass conservation law we have

$$\rho_r v_r S_0 = \rho_{\scriptscriptstyle B} v_{\scriptscriptstyle B} S.$$
(4.20)

On the contact-discontinuity line we have

$$\left. \begin{array}{l} p_6 = p_{\scriptscriptstyle B}, \\ v_6 = v_{\scriptscriptstyle B}. \end{array} \right\}$$
(4.21)

The escape of gas from the narrow channel into a wide one occurs adiabatically, and consequently

$$\frac{p_{\scriptscriptstyle B}}{p_r} = \left(\frac{\rho_{\scriptscriptstyle B}}{\rho_r} \right)^k.$$
(4.22)

Thus

$$\frac{v_r}{v_6} \left(\frac{p_r}{p_6} \right)^{\frac{1}{k}} = \frac{S}{S_0}.$$
(4.23)

Equation (4.23) is a solution of the problem formulated. It yields the connection between the quantity Δp_b and the section S/S_0 for a specified pressure at the inlet to the expander Δp_d, and also the dependence of Δp_b on Δp_d for a specified ratio of the expander cross section to the inlet channel S_0.

Inasmuch as the values of the velocities $\underline{v}$ are uniquely determined by the pressure on the front $\underline{p}$, we can plot the function $vp^{1/k} = f(p)$ (Fig. 86).

With the aid of the plot (Fig. 86) the coefficient of extinction of the expansion chamber is calculated in the following sequence:

1) the specified pressure p_d at the inlet to the expander is used
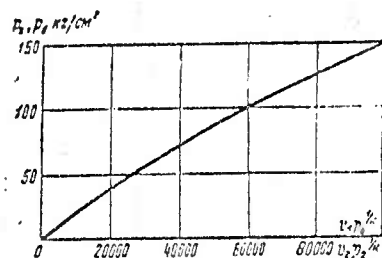
Fig. 86. Auxiliary plot for the calculation of the extinction coefficient of an expansion chamber.
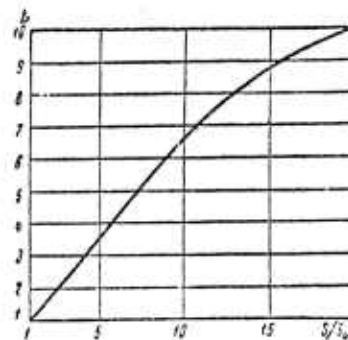
Fig. 87. Extinction coefficient of expansion chambers.

to determine $v_d p_d^{1/k} = f(p_d)$;

2) the obtained value is divided by the ratio of the cross section areas S/S_0 and $v_b p_b^{1/k} = v_d p_d^{1/k} S_0/S$ determined; from this value (also with the aid of Fig. 86) one calculates p_b and the excess pressure $\Delta p_b = p_b - p_a$;

3) in order to change over to the pressure in the channel behind the expander, $\Delta p'_b$, it is necessary to take into account the transition of the wave from the broad tube into the narrow one, which is characterized by the coefficient $\Omega = \Delta p_b / \Delta p'_b$. This coefficient is picked off the plot of Fig. 83;

4) the extinction coefficient of the expansion chamber is calculated as the ratio of the initial pressure to the pressure behind the expansion chamber $\gamma = \Delta p_d / \Delta p'_b$.

For approximate estimates one can use an even simpler relation. Calculations show that approximately

$$\gamma_p \simeq \frac{\Delta p_{\text{нач}}}{\Delta p_{\text{кон}}} \simeq \left(\frac{S}{S_0}\right)^{0.8}. \qquad (4.24)$$

The dependence of the extinction coefficient on the ratio of the cross section areas of the channel and expander is shown in Fig. 87.

Using this plot, it is possible to determine immediately the sought coefficient γ_r, without resorting to intermediate derivations.

It must be borne in mind, however, that although γ_r is practically independent of Δp_{nach}, the extinction coefficient γ is connected with this quantity via the coefficient Ω.

The length of the expansion chamber should be approximately three times larger than the diameter. A further increase of the length is ineffective.

§4. PRINCIPLES OF THE DESIGN OF STRUCTURES TO WITHSTAND THE ACTION OF AN AERIAL SHOCK WAVE

In choosing the scheme for designing a structure to withstand the action of a shock wave, great importance is attached to the ratio of the period of the natural oscillations of the structure T_0 to the length of the compression phase τ_+.

If the ratio is $\tau_+/T_0 \ll 1$, the design is based on impulse; if $\tau_+/T_0 \gg 1$ it is based on maximum pressure. When the quantities τ_+ and T_0 are of the same order of magnitude, it is necessary to take into account the character of the time variation of the pressure.

B.A. Olisov has shown that when the load is approximated by the linear relation $F = F_m(1 - t/\tau)$, its replacement by the instantaneous impulse $S = \int_0^{\tau_+} F(t)dt$ leads to an exaggeration of the calculated values of the strains and stresses; this exaggeration is the larger, the greater the ratio τ_+/T_0. Thus, when $\tau_+/T_0 = 0.33$ the error in estimating the strains amounts to +12%, when $\tau_+/T_0 \simeq 0.25$ it is approximately +7%, and when $\tau_+/T_0 = 0.16$ it is less than +3%.

It follows therefore that the assumption that the action of the shock wave in an aerial explosion has an impulse character can be assumed as practically acceptable if $\tau_+ \leq 0.25\ T_0$.

The "static" character of the action of the shock wave should be

observed according to the theory when τ_+ exceeds T_0 by a factor of approximately 10.

In Table 20 are listed the values of the natural periods and destructive loads of typical structural elements.

TABLE 20

1 Конструкция	2 Кирпичные стенки		5 Железо-бетонная стена 0,25 м	6 Пере-крытия по дере-вянным балкам	7 Легкие перего-родки	8 Застек-ление
	3 два кирпича	4 полтора кирпича				
9 Период собствен-ных колебаний, T_0 сек	0,01	0,015	0,015	0,3	0,07	0,02—0,04
10 Статическая на-грузка, F кг/см²	0,45	0,25	3,0	0,10—0,16	0,05	0,05—0,10
11 Импульсная на-грузка, S кгсек/м²	220	190	—	—	—	—

1) Structure; 2) brick walls; 3) two bricks; 4) one and a half bricks; 5) reinforced-concrete walls 0.25 m; 6) decks supported by wooden beams; 7) light partitions; 8) glazed structure; 9) period of natural oscillations, T_0, sec; 10) static load, F, kg/cm^2; 11) impulse load, S, kg-sec/m^2.

Comparing the natural period of the structure with the time of action of the positive phase of the shock wave, it is easy to establish whether any of the indicated computation schemes can be employed. In the case of explosion of charges of ordinary explosive substances, cases of purely impulsive or purely static action of the load are exceptions. It then becomes necessary to take into account the time variation of the pressure. The approximate construction of the pressure pattern for partitions of finite dimensions is then carried out by the method indicated in §2.

As regards the structure, it is reduced as a rule, with the aid of the Lagrange method, to a dynamic system with one or several degrees of freedom.*

For simplicity let us consider the computation scheme for an elastic system with one degree of freedom.

The differential equation of motion of such a system in the time interval $0 < t < t_{obt}$ will in accordance with (4.88) be

$$M\ddot{z} + kz = \Delta p_{\text{отр}}\left(1 - \frac{t}{2t_{обт}}\right),$$ (4.25)

where M is the reduced mass; z is the displacement of the reduction point; k is the reduced stiffness coefficient; and t_{obt} is the time of flow-around (the travel time of the sound wave along the smaller of the dimensions of the structure).

Its general solution is

$$z = c_1 \sin \omega t + c_2 \cos \omega t + \frac{\Delta p}{M\omega^2}\left(1 - \frac{t}{2t_{обт}}\right),$$ (4.26)

or, after determining the arbitrary constants from the initial conditions ($z = 0$ and $\dot{z} = 0$ when $t = 0$),

$$z = \frac{\Delta p}{M\omega^2}\left\{1 - \cos \omega t - \frac{1}{2t_{обт}}\left[t - \frac{\sin \omega t}{\omega}\right]\right\}.$$ (4.27)

The velocity of motion of the reduction point is

$$\dot{z} = \frac{\Delta p}{M\omega^2}\left\{\omega \sin \omega t - \frac{1}{2t_{обт}}(1 - \cos \omega t)\right\}.$$ (4.28)

In Formulas (4.26)-(4.28) ω denotes the natural frequency of the system

$$\omega^2 = k/M.$$ (4.29)

Starting with the instant of time $t = t_{obt}$, the value of the load varies as (4.28).

The motion is determined by the differential equation

$$M\ddot{z} + kz = \Delta p^*,$$ (4.30)

which is integrated under the following initial conditions: at the instant $t = t_{obt}$ we have

$$\left.\begin{array}{l} z = z_1, \\ \dot{z} = \dot{z}_1. \end{array}\right\}$$ (4.31)

The solution of Eq. (4.30) will be

$$z = z_1 \cos \omega (t - t_{o6\tau}) + \frac{\dot{z}_1}{\omega} \sin \omega (t - t_{o6\tau}) +$$

$$+ \frac{\Delta p}{M\omega^2} \left\{ 1 - \cos \omega (t - t_{o6\tau}) + \frac{1}{\tau_+ - t_{o\tau p}} \frac{1}{\omega} \sin \omega (t - t_{o6\tau}) - \frac{t - t_{o\tau p}}{\tau_+ - t_{o\tau p}} \right\};$$

$$(4.32)$$

$$\dot{z} = - z_1 \omega \sin \omega (t - t_{o6\tau}) + \dot{z}_1 \cos \omega (t - t_{o6\tau}) -$$

$$- \frac{\Delta p}{M\omega^2} \left\{ 1 + \omega \sin \omega (t - t_{o6\tau}) - \frac{1}{\tau_+ - t_{o\tau p}} [1 - \cos \omega (t - t_{o6\tau})] \right\}.$$

$$(4.33)$$

Putting $\dot{z} = 0$ and calculating from (4.33) the corresponding values of t*, we can readily obtain the maximum displacement z_{max} from Relation (4.32) where we substitute $t = t*$.

The displacement z_{max} of the reduction point uniquely determines the maximum strains and stresses of the system.

In principally the same manner one solves more complicated problems in the dynamic design of systems that are deformed in the elasto-plastic zone.

Example. Establish the limiting TNT charge that can be exploded without danger of breaking the glass windows in a house located 1000 m from the planned center of explosion.

Solution. On the basis of the data of Table 20, the static load that the windows can withstand is approximately 0.05 kg/cm^2.

Since this pressure is equal to the reflection pressure, we assume for the direct wave, with some margin, $\Delta p_f = 0.02$ kg/cm^2.

In accordance with Sadovskiy's formula for an explosion on the earth's surface

$$\Delta p = 0.95 \frac{\sqrt[3]{G}}{r} + 3.9 \frac{\sqrt[3]{G^2}}{r^2} + 13.0 \frac{G}{r^3}.$$

Putting $\Delta p = 0.02$ and solving this equation with respect to $\sqrt[3]{G}/r$, we obtain

$$\frac{\sqrt[3]{G}}{r} \simeq 0.018.$$

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From this we obtain for the specified value $r = 1000$ m the sought weight of the charge

$$G = 5800 \text{ kg.}$$

The time of action of the shock wave will be

$$t_q = 1.7 \cdot 10^{-3} \sqrt[6]{G} \sqrt{r} = 1.7 \cdot 10^{-3} \sqrt[6]{5800} \sqrt{1000} =$$
$$= 1.7 \cdot 10^{-3} \cdot 4.2 \cdot 31.6 = 0.226 \text{ sec.}$$

This time is approximately 10 times greater than the natural period of the glass windows.

Consequently, the estimating scheme employed is perfectly acceptable.

§5. SIMPLEST DIFFRACTION PROBLEMS IN THE THEORY OF UNDERWATER EXPLOSIONS

Let us consider first the simplest case of the diffraction of a shock wave around an absolutely rigid disk.

We shall solve the problem in the linear formulation. To estimate the hydrodynamic fields, we use the radiation integral.

The potential characterizing the propagation of the direct wave, in the case of motion with spherical symmetry, can be represented, as is well known, in the form

$$\varphi = -\frac{\psi(\tau - R)}{R} \sigma_0(\tau - R), \qquad (4.34)$$

where for the sake of convenience R denotes the dimensionless distance $R = r/R_{03}$, while τ stands for the dimensionless time

$$\tau = \frac{t a_0}{R_{03}}. \qquad (4.35)$$

The value of the normal component of the velocity in the partition plane will be (Fig. 88)

$$v_n = \frac{\partial \varphi}{\partial n} = \frac{\partial \varphi}{\partial R} \frac{\partial R}{\partial n} =$$
$$= -\frac{z_0}{R} \left\{ \frac{\psi(\tau - R)\sigma_0(\tau - R)}{R^2} - \frac{\frac{d}{dR}[\psi(\tau - R)\sigma_0(\tau - R)]}{R} \right\}. \qquad (4.36)$$

The boundary condition will be satisfied if one places at the
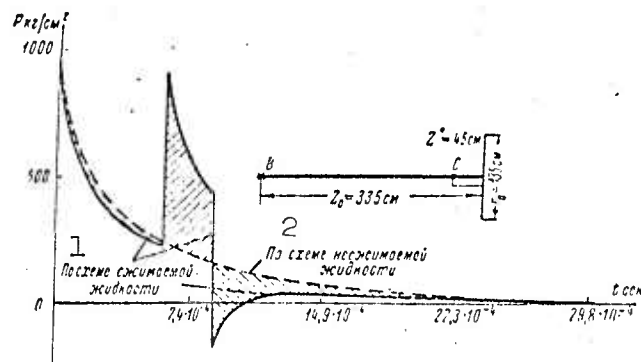
Fig. 88. Comparison of the character of variation of the pressure in the case of an explosion, with allowance for the presence of a rigid disk, using the incompressible liquid (dash) and compressible liquid (solid) schemes. B) Point of explosion of a TNT charge weighing 75 kg; C) measurement point. 1) Using the compressible liquid scheme; 2) using the incompressible liquid scheme.

points of the disk hydrodynamic sources such that the flow velocities due to these sources turn out to be equal and opposite to the velocity determined by Relation (4.36).

As already mentioned, such a result is attained with the aid of the radiation integral, which in this case is conveniently written in the form

$$\varphi_A = -\frac{1}{2\pi} \int\limits_0^{2\pi} \int\limits_0^{r_0} \frac{v_n(\tau - R^*)}{R^*} r\,dr\,d\varphi,$$ (4.37)

where R^* is the distance from the point of the disk to the point C (see Fig. 77).

Considering the case of axial symmetry, we obtain on the basis of (4.36)

$$\varphi_A = z_0 \int\limits_0^{r_0} \frac{\psi(\tau - R - R^*)}{R^2 R^*} \sigma_0'(\tau - R - R^*)\,r\,dr \ -$$

$$- z_0 \int\limits_0^{r_0} \frac{\frac{d}{d(R+R^*)}[\psi(\tau - R - R^*)\sigma_0(\tau - R - R^*)]}{R^2 R^*} \ r\,dr.$$ (4.38)

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We introduce the substitution

$$R + R^* = x,$$

where

$$R = \sqrt{r^2 + z_0^2};$$
$$R^* = \sqrt{r^2 + z^{*2}},$$

then

$$\left(\frac{r}{R} + \frac{r}{R^*}\right) dr = \frac{rx\, dr}{RR^*} = dx;$$

$$\frac{rdr}{RR^*} = \frac{1}{x}\, dx; \qquad \frac{1}{x} = \frac{1}{R + R^*} = \frac{R - R^*}{z_0^2 - z^{*2}};$$

$$rdr = RR^* \frac{R - R^*}{z_0^2 - z^{*2}}\, dx;$$

$$\frac{z_0^2 - z^{*2}}{x} = R - R^*; \qquad \frac{z_0^2 - z^{*2}}{x} + x = 2R;$$

$$z_0^2 - z^{*2} + x^2 = 2Rx,$$

$$(z_0^2 - z^{*2} + x^2)^2 = 4R^2x^2;$$

$$R^2 = \frac{(z_0^2 - z^{*2} + x^2)^2}{4x^2};$$

$$I_0 = \int_0^{r_0} \frac{\psi(\tau - R - R^*)}{R^2 R^{*2}} \sigma_0(\tau - R - R^*)\, rdr =$$

$$= \int_{z_0 + z^*}^{R_0 + R^*} \frac{\psi(\tau - x)}{R^2 R^*} \sigma_0(\tau - x) RR^* \frac{dx}{x} = \int_{z_0 + z^*}^{R_0 + R^*} \frac{\psi(\tau - x)}{R^2} \sigma_0(\tau - x) \frac{dx}{x} =$$

$$= 4 \int_{z_0 + z^*}^{R_0 + R^*} \frac{\psi(\tau - x)\, \sigma_0(\tau - x)}{(z_0^2 - z^{*2} + x^2)} x\, dx. \qquad (4.39)$$

In perfect analogy

$$I_1 = \int_0^{r_0} \frac{\dfrac{d}{d(R^* + R)}[\psi(\tau - R - R^*)\sigma_0(\tau - R - R^*)]}{R^2 R^{*2}}\, rdr =$$

$$= 2 \int_{z_0 + z^*}^{R_0 + R_0^*} \frac{\dfrac{d}{dx}[\psi(\tau - x)\sigma_0(\tau - x)]\, dx}{z_0^2 - z^{*2} + x^2}. \qquad (4.40)$$

Thus, the potential at the point C will be

$$\varphi_c = 4 \int_{z_0 + z^*}^{R_0 + R^*} \frac{\psi(\tau - x)\sigma_0(\tau - x)}{(z_0^2 - z^{*2} + x^2)^2} x\, dx -$$

$$-2 \int_{z_\bullet+z^\bullet}^{R_\bullet+R_0^\bullet} \frac{\frac{d}{dx}\left[\psi\left(\tau-x\right)\sigma_0\left(\tau-x\right)\right]}{z_0^2-z^{\ast\prime}+x^2}\,dx.$$

Integrating by parts, we get

$$\varphi_c = 2\,\frac{\psi\left(\tau-x\right)\sigma_0\left(\tau-x\right)}{z_0^2-z^{\ast\prime}+x^2}\,\bigg|_{z_\bullet+z^\bullet}^{R_\bullet+R_0^\bullet}, \tag{4.41}$$

or, returning to the previous notation,

$$\varphi_c = 2z_0\left\{\frac{\psi\left(\tau-z_0-z^*\right)\sigma_0\left(\tau-z_0-z^*\right)}{2z_0\left(z_0+z^*\right)}-\right.$$

$$-\frac{\psi\left(\tau-V\overline{r_0^2+z_0^2}-V\overline{r_0^2+z^{\ast\prime}}\right)}{2V\overline{r_0^2+z_0^2}\left(V\overline{r_0^2+z_0^2}+V\overline{r_0^2+z^{\ast\prime}}\right)}\,\sigma_0\left(\tau-V\overline{r_0^2+z_0^2}-\right.$$

$$\left.\left.-V\overline{r_0^2+z^{\ast\prime}}\right)\right\}. \tag{4.42}$$

If we add to the potential (4.42) the potential of the direct wave and use the well-known relation $p = \rho_0(\partial\varphi/\partial\tau)$, then we obtain the values of the pressures at the point lying on the symmetry axis:

$$p\left(0,z^*\right) = \rho_0\,\frac{1}{z_0-z^*}\,\frac{\partial}{\partial\tau}\left[\psi\left(\tau-z_0+z^*\right)\sigma_0\left(\tau-z_0+z^*\right)\right]+$$

$$+\rho_0\,\frac{1}{z_0+z^*}\,\frac{\partial}{\partial\tau}\left[\psi\left(\tau-z_0-z^*\right)\sigma_0\left(\tau-z_0-z^*\right)\right]-$$

$$-\rho_0\,\frac{z_0}{V\overline{r_0^2+z_0^2}\left(V\overline{r_0^2+z_0^2}+V\overline{r_0^2+z^{\ast\prime}}\right)}\,\frac{\partial}{\partial\tau}\left[\psi\left(\tau-V\overline{r_{0,}^2+z_0^2}-\right.\right.$$

$$\left.\left.-V\overline{r_0^2+z^{\ast\prime}}\right)\cdot\sigma_0\left(\tau-V\overline{r_0^2+z_0^2}-V\overline{r_0^2+z^{\ast\prime}}\right)\right]. \tag{4.43}$$

We see that the first two terms of (4.43) are the same as if the partition were to be infinite. They determine the direct and reflected waves.

The third term determines the rarefaction waves traveling toward the point from the edges of the disk and due to the increased velocity of the flow of liquid through the edge of the partition. The wave pattern in the case of diffraction around a disk is shown for the sake of clarity in Fig. 88.

We can obtain analogously the pressure at the points of the symmetry axis, located behind the disk. It will be determined by the equa-

tion

$$p(0, z^*) = \frac{\rho_0 z_0}{\sqrt{r_0^2 + z_0^2}\left(\sqrt{r_0^2 + z_0^2} + \sqrt{r_0^2 + z^{*1}}\right)} \frac{\partial}{\partial \tau} \psi\left(\tau - \sqrt{r_0^2 + z_0^2} - \right.$$
$$\left. - \sqrt{r_0^2 + z^{*1}}\right) \sigma_0 \left(\tau - \sqrt{r_0^2 + z_0^2} - \sqrt{r_0^2 + z^{*1}}\right). \qquad (4.44) \qquad (4.44)$$

It is of interest to solve the same problem under the assumption of incompressibility of the medium.

The final formulas can in this case be obtained both as a result of going to the limit to infinite velocity of sound in (4.43) and (4.44), and by direct examination of the potential

$$\varphi_1 = - \frac{\psi(\tau)}{R}. \qquad (4.45)$$

Under the foregoing assumptions the radiation integral (4.37) is rewritten in the form

$$\varphi_A = - \frac{1}{2\pi} \int_0^{2\pi} \int_0^{r_*} \frac{z_0 \psi(\tau)}{R^4 R^*} r \, dr \, d\varphi. \qquad (4.46)$$

This integral can be obtained in closed form if the point A lies on the axis $z = 0$.

After simple transformations we obtain

$$\varphi(0, z^*) = \psi(\tau) \frac{\sqrt{z_0^2 + r_0^2} - z_0 \sqrt{r_0^2 + z^{*1}}}{(z^{*1} - z_0^2)\left(\sqrt{z_0^2 + r_0^2}\right)}. \qquad (4.47)$$

Recognizing that $p = \rho_0(\partial\varphi/\partial\tau)$, we arrive at the conclusion that the presence of a partition of finite dimensions in the liquid leads to a proportional change in the values of the pressure at the specified point. The proportionality coefficient depends only on the geometrical characteristics of the problem.

Figure 88 shows a comparison of the character of variation of the pressure in the case of an explosion, with allowance for the presence of a rigid disk, using the incompressible and compressible liquid schemes. We see that the resultant impulse is the same in both cases. The assumption that the medium is incompressible, in full accordance

with the physical picture of the phenomenon, gives only a certain re-distribution of the pressures. This circumstance makes it possible in individual cases to employ, when the resultant impulse is of decisive significance, the simpler incompressible-liquid scheme for the approximate calculations.

An estimate of the values of the pressures at an arbitrary point of space entails much more serious difficulties. Such a problem has been solved so far only for a plane wave.

Without dwelling on the derivations and transformations, which are essentially close to those presented above, we shall formulate only the final conclusions:

1) in the case of the flow of a shock wave around a disk, a rare-faction wave is produced, propagates from its edges, and distorts the pattern of the direct wave;

2) at points situated in front of the disk and having projections on it, the portion of the pressure pattern, containing the fronts of the direct and reflected waves, remains the same as in the case of an infinite partition. The diffraction distorts only the remaining part of the pattern, reducing the pressure;

3) when a direct shock wave flows in behind a disk, its front collapses and the maximum pressure at the points behind the disk greatly decreases, with the exception of the points lying on the symmetry axis of the disk.

When the diffraction wave arrives at these points, the curvature of its front greatly increases and a pressure jump again appears on the symmetry axis, of a size close to the maximum amplitude of the direct wave;

4) the pressures at points that had no projections on the disk are distorted by the positive and negative waves emerging from the

surface of the disk and from its edge;

5) analogous phenomena should occur also in the case of the diffraction of a direct shock wave around flat partitions of other forms.

Fig. 89. Pressure patterns at points located ahead of the disk.
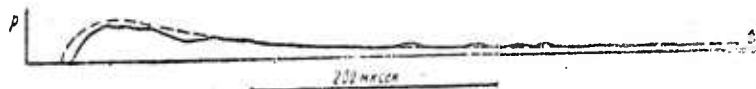
Fig. 90. Pressure patterns at points located behind the disk.

The experimental data confirm the possibility of estimating the diffraction fields with the aid of the radiation integral.

By way of illustration, Fig. 89 shows the pressure patterns at points located ahead of the disk, while Fig. 90 shows the patterns behind the disk. In both cases good agreement is observed between the theory (dashed lines) and experiment (solid lines).

However, the method developed here is not always the most effec-

tive for the study of diffraction fields. It is possible to use for this purpose other methods, which we now proceed to develop.

Let us consider the following problem.*.Assume that an absolutely rigid stationary plate of infinite length and zero thickness is immersed in water to a depth T. The lower edge of the plate is parallel to the free surface of the water.

A plane shock wave of rectangular form with unit amplitude and with time of action τ is incident on the plane of the partition at an angle θ. When the direct wave reaches the lower edge of the partition, a diffraction field is produced, and at each point of space around the partition there is produced a pressure whose duration of action depends both on the value of τ and on the time when the wave reflected from the free surface arrives there.

It is required to determine the distribution and the time variation of the pressures on the partition.

We shall first consider a wave of unit amplitude and of infinite time of action. We assume that the front of the direct wave is parallel to the edge of the partition and arrives at the latter at the instant of time $t = 0$. Then for the instant of time $t > 0$ the pattern shown in Fig. 91 will be valid, where OE stands for the partition, NE and AB for the fronts of the direct wave at different instants of time, CD for the front of the transmitted wave, EF for the front of the reflected wave, and KDGFK for the diffraction region.

By definition, a unit wave is one with unity pressure behind its front and zero pressure ahead of its front. Therefore the pressure is equal to zero ahead of the fronts CD and NE and outside the arc DK, and is equal to unity behind the front CD and outside the line DGFEN.

In the region EKF the pressure is doubled, since the wave reflected from the partition is superimposed here on the direct wave.
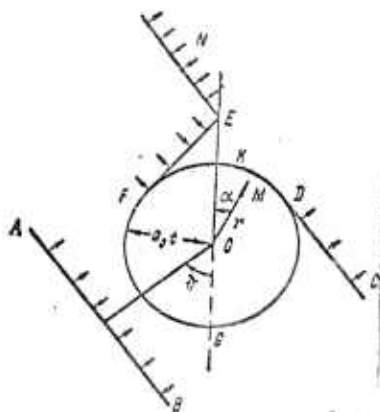
Fig. 91. Pattern of
diffraction of a shock
wave around the edge
of a semi-infinite
partition.

By virtue of the assumed absolute stiffness of the partition, the boundary condition on it should be

$$v_n = \frac{\partial \varphi}{\partial n} = 0.$$

We shall find it convenient to use this condition in a different notation.

Since $\partial \varphi / \partial n = 0$ on the screen and since in accordance with the Lagrange equation

$$p = \rho_0 \frac{\partial \varphi}{\partial t},$$

from which

$$\frac{\partial p}{\partial n} = \rho_0 \frac{\partial}{\partial n}\left[\frac{\partial \varphi}{\partial t}\right] = \rho_0 \frac{\partial}{\partial t}\left[\frac{\partial \varphi}{\partial n}\right],$$

we should have on the screen

$$\left(\frac{\partial p}{\partial n}\right)_{\text{экр}} = 0. \tag{4.48}$$

Thus, the problem reduces to the solution of the wave equation

$$\frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r} + \frac{1}{r^2}\frac{\partial^2 p}{\partial a^2} - \frac{1}{a_0^2}\frac{\partial^2 p}{\partial t^2} = 0 \tag{4.49}$$

in a circle of radius $a_0 t$ under the boundary conditions

$$p(a_0, t, a) = \begin{cases} 0 & 0 \leqslant a < \theta \\ 1 & \theta < a < 2\pi - \theta \\ 2 & 2\pi - \theta < a \leqslant 2\pi \end{cases} \tag{4.50}$$

$\partial p / \partial n = 0$ on both edges of the cut OK.

We present directly the solution of (4.49) with boundary conditions (4.50):*

$$p(r, a) = 1 \pm \frac{1}{\pi} \arctan \frac{2\sqrt{2}\sin\frac{\theta}{2}\cos\frac{a}{2}\sqrt{\frac{a_0 t}{r} - 1}}{\frac{a_0 t}{r} - 1 + \cos\theta - \cos a}. \tag{4.51}$$

The minus sign corresponds to the points behind the plane of the partition, and the plus sign to the points ahead of it.

We put

$$f(\theta,\alpha,r,t) = \frac{1}{\pi} \operatorname{arc\,tg} \frac{2\sqrt{2}\sin\frac{\theta}{2}\cos\frac{\alpha}{2}\sqrt{\frac{a_0 t}{r}-1}}{\frac{a_0 t}{r}-1+\cos\theta-\cos\alpha}; \tag{4.52}$$

then the pressure at the point behind the partition, in accordance with Fig. 91, will be

$$p_{\mathrm{I}} = \left\{ \sigma_0 \left[t - \frac{r}{a_0}\cos(\alpha'-\theta) \right] - \sigma_0\left(t-\frac{r}{a_0}\right) \right\} \sigma_0(\alpha'-\theta) +$$
$$+ \left[1 - f(\theta,\alpha,r,t)\right] \sigma_0\left(t-\frac{r}{a_0}\right)\left[\sigma_0(\alpha')-\sigma_0(\alpha'-\pi)\right], \tag{4.53}$$

where the first and second terms determine the pressure at the point before and after the arrival of the diffraction wave, respectively.

We furthermore denote by α' the angle of the investigated point, measured clockwise from the partition.

For points behind the plane of the partition α' = α, while for points ahead of it α' = 2π − α.

The pressure p_{II} at the point ahead of the plane of the partition will be

$$p_{\mathrm{II}} = \left\{ \sigma_0\left[t - \frac{r}{a_0}\cos(\alpha'-\theta) \right] - \sigma_0\left(t-\frac{r}{a_0}\right) \right\} \sigma_0(\alpha'-\pi) +$$
$$+ \left\{ \sigma_0\left[t - \frac{r}{a_0}\cos(\alpha'-2\pi+\theta) \right] - \sigma_0\left(t-\frac{r}{a_0}\right) \right\} \sigma_0(\alpha'-2\pi+\theta) +$$
$$+ \left[1 + f(\theta,\alpha,r,t)\right] \sigma_0\left(t-\frac{r}{a_0}\right)\sigma_0(\alpha'-\pi). \tag{4.54}$$

The first term of this formula determines the action of the direct wave, while the second determines the action of the reflected wave and the third determines the pressure at the point after the diffraction wave arrives at it.

Combining the expressions for p_{I} and p_{II} we obtain a formula for the determination of the pressure at any point near the screen, for the propagation of a wave of unit amplitude and of infinite time of action:

$$p_1(t) = \left[1 - f(\theta,\alpha,r,t)\right] \sigma_0\left(t-\frac{r}{a_0}\right)\left[\sigma_0(\alpha')-\sigma_0(\alpha'-\pi)\right] +$$

$$+ \left[1 + f(\theta, a, r, t)\right] \sigma_0 \left(t - \frac{r}{a_0}\right) \sigma_0 (a' - \pi) +$$
$$+ \left[\sigma_0\left(t - \frac{r}{a_0} \cos(a' - \theta)\right) - \sigma_0\left(t - \frac{r}{a_0}\right)\right] \sigma_0 (a' - \theta) +$$
$$+ \left[\sigma_0\left(t - \frac{r}{a_0} \cos(a' - 2\pi + \theta)\right) - \right.$$
$$\left. - \sigma_0\left(t - \frac{r}{a_0}\right)\right] \sigma_0 (a' - 2\pi + \theta). \qquad (4.55)$$

To abbreviate the notation we put

$$p_1(t) = F(\theta, a, r, t). \qquad (4.56)$$

It is obvious that to solve the problem of the diffraction of a rectangular wave of duration τ around the edge of a semi-infinite screen it is necessary to add to Solution (4.56) also the solution of the problem of the diffraction of the unit rarefaction wave (Fig. 92).

Such a solution can be written in the form

$$p_2(t) = -F(\theta, a, r, t - \tau). \qquad (4.57)$$

Thus, if we disregard the influence of the free surface near the partition, the sought field will be

$$p(t) = F(\theta, a, r, t) - F(\theta, a, r, t - \tau). \qquad (4.58)$$

Going over to an account of the influence of the free surface, we shall assume the diffraction waves to originate at the point O. Then, using the method of mirror images of the sources and sinks, we obtain without difficulty (see Fig. 92):

$$p_3(t) = -F(\theta, a_1, r_1, t) + F(\theta, a_1, r_1, t - \tau). \qquad (4.59)$$

The sought solution will assume the form

$$p(t) = F(\theta, a, r, t) - F(\theta, a, r, t - \tau) -$$
$$- F(\theta, a_1, r_1, t) + F(\theta, a_1, r_1, t - \tau). \qquad (4.60)$$

A generalization of the obtained solution to include a wave of arbitrary form can be effected with the aid of the Duhamel integral.

An analysis of the foregoing relations enables us to draw the following conclusions:

1) the diffraction field reduces the pressure at the points behind the partition as compared with the pressure in the direct wave.

The smaller the angle α', the more considerable this reduction;

2) at the points determined by the angles $\alpha' < \theta$, the pressure jump vanishes and is replaced by a gradual increase in pressure.

As the angle α' increases from zero to θ, the increase in pressure becomes steeper and steeper, and when $\alpha' = \theta$ a pressure jump appears, which is retained with further increase in the angle α';
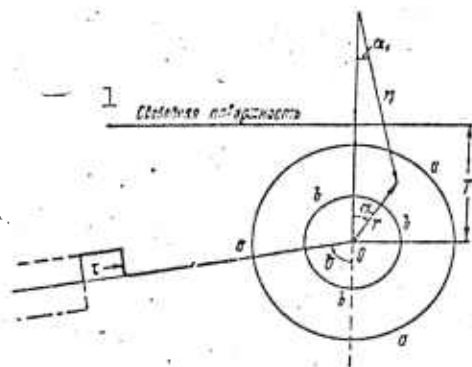
Fig. 92. Diagram of diffraction of underwater wave at edge of plate suspended in water. 1) Free surface.

3) the time of action of the pressure at the points behind the partition can be larger than in the direct wave. This is explained by the fact that the partition excludes the direct influence of the free surface.

In spite of the highly schematic representation of the phenomena, the formulas presented can be used for tentative practical estimates.

We note further that A. N. Patrashev developed an energy method for solving diffraction problems in the nonlinear formulation. Its practical utilization, however, presents considerable difficulties.

§6. HYDRODYNAMIC FIELD DUE TO TRANSLATIONAL DISPLACEMENT OF A PLATE. GENERALIZATION OF THE CONCEPT OF APPARENT MASS

In addition to the diffraction problems considered above in most general outlines, serious difficulties in the estimates of the external forces arise in connection with the need for taking into account the variation of the pressure resulting from the displacement of the structure itself.

A study of the forces acting on a solid that moves in unsteady motion is carried out in hydromechanics, as is well known, with the aid of the apparent masses. The liquid is then regarded as incompress-

ible. In the problems under consideration, such an assumption, however, cannot be regarded as fully acceptable: it is necessary to take into account here the compressibility of the medium and the finite propagation velocities of the disturbances.

Let us consider such a problem using as the simplest example the translational motion of a round disk with velocity $\dot{z}(t)$.

In accordance with Eq. (4.37), the potential at an arbitrary point of the disk is determined by the relation

$$\varphi_A(R, t) = \frac{1}{2\pi} \int\limits_0^{2\pi} \int\limits_0^a \frac{\frac{d}{dt}\left[z\left(t - \frac{R}{a_0}\right)\sigma_0\left(t - \frac{R}{a_0}\right)\right]}{R} \, r\,d\psi\,dR, \tag{4.61}$$

where R is the distance from the point A to the disk and $\underline{a}$ is the radius of the disk.

The value of the pressure at this same point will be

$$p(R, t) = \rho_0 \frac{\partial \varphi}{\partial t}. \tag{4.62}$$

The resultant hydrodynamic force is determined from the relation

$$F = 2\pi \int\limits_0^a pr\,dr. \tag{4.63}$$

After rather cumbersome derivations and transformations, we obtain

$$F = 2a^2 a_0 \rho_0 \left\{ \frac{\pi}{2} \dot{z}(t)\sigma_0(t) - \frac{a_0}{a} z(t)\sigma_0'(t) + \right.$$

$$\left. + \frac{a_0}{a} \int\limits_0^{2\pi} z'\left(t - \frac{2a}{a_0}\sin\psi\right)\sigma_0\left(t - 2\frac{a}{a_0}\sin\psi\right)\sin\psi\,d\psi \right\} \tag{4.64}$$

Taking the transition to the limit, we can show that when the velocity of sound is infinite we have

$$\lim_{a_0 \to \infty} F = \frac{8}{3} \rho_0 a^3 \ddot{z}(t), \tag{4.65}$$

which coincides with the well-known result characterizing the value of the apparent mass in the translational displacement of a disk in the direction normal to its surface.

The relation (4.64) can be represented in a different form by ex-

panding the function $z\left(t-2\frac{a}{a_0}\sin\psi\right)$ in a Taylor series about $\underline{t}$.

Since

$$z\left(t-\frac{2a}{a_0}\sin\psi\right)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left(2\frac{a}{a_0}\sin\psi\right)^n\frac{d^nz(t)}{dt^n},$$

we have

$$F=2a^2a_0\rho_0\left\{\frac{\pi}{2}\dot{z}(t)\,\sigma_0(t)-\frac{a_0}{a}z(t)\,\sigma_0(t)+\right.$$

$$+\frac{a_0}{a}\int_0^{\varphi_1}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left(\frac{2a}{a_0}\sin\psi\right)^n\frac{d^nz(t)}{dt^n}\sin\psi d\psi\Big\}+$$

$$+2aa_0^2\rho_0\sum_{n=2}^{\infty}\frac{(-1)^n}{n!}\left(\frac{2a}{a_0}\right)^n\frac{d^nz(t)}{dt^n}2^n\frac{\left[\Gamma\left(\frac{n}{2}+1\right)\right]^2}{\Gamma(n+2)}\sigma_0\left(t-\frac{2a}{a_0}\right),\qquad(4.66)$$

where

$$\varphi_1=\arcsin\frac{ta_0}{2a}\left[\sigma_0(t)-\sigma_0\left(t-2\frac{a}{a_0}\right)\right];\qquad(4.67)$$

$\Gamma(n)$ is the gamma function.

As is well known, if $\underline{n}$ is an integer we have

$$\Gamma(n)=n!$$

$$\Gamma\left(n+\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2^n}(2n-1)!!$$

Consequently, the last term of (4.66) can be readily calculated.

It can be shown that as soon as the product ta_0 exceeds $2a$, the generalized force becomes approximately equal to

$$F=\frac{8}{3}\rho_0a^3\ddot{z}(t).$$

It follows therefore that the unsteady process, in the study of which it is necessary to take into account the finite velocities of propagation of the disturbances, has in the present case a duration equal to twice the travel time of the sound wave from the center to the edge of the disk. After the lapse of this time interval, an account of the influence of the displacement of the partition on the hydrodynamic fields should be carried out in the same way as for the case of an incompressible liquid, i.e., by introducing the apparent-

mass coefficient.

Calculations show that during the first period of motion ($t <$ $< 2a/a_0$), one can confine oneself with good approximation to an account of only three terms of the series ($t < 2a/a_0$), so that the problem of the displacement of a body under the influence of a shock wave changes from a problem involving integration of integral-differential equations into a problem of integration of ordinary differential equations.

The resultant hydrodynamic force is determined, accurate to third-order derivatives, on the basis of Formulas (4.66)-(4.67), by the relation

$$F = \pi a^2 \left\{ k_z \frac{z}{a} \rho_0 a_0^2 + k_z' \rho_0 a_0 \dot{z} + k_z'' a \ddot{z} + \right.$$
$$\left. + k_z''' \frac{\rho_0}{a_0} a^2 \dddot{z} \right\} \left[\sigma_0(t) - \sigma_0 \left(t - 2 \frac{a}{a_0} \right) \right] +$$
$$+ \frac{8}{3} \rho_0 a^3 \ddot{z} \sigma_0 \left(t - 2 \frac{a}{a_0} \right), \qquad (4.68)$$

where

$$k_z = \frac{2}{\pi} \sqrt{1 - \left(\frac{t a_0}{2a} \right)^2}; \qquad (4.69)$$

$$k_z' = 1 - \frac{2}{\pi} \arcsin \frac{t a_0}{2a} + \frac{1}{\pi} \frac{t a_0}{a} \sqrt{1 - \left(\frac{t a_0}{a} \right)^2}; \qquad (4.70)$$

$$k_z'' = \frac{8}{3\pi} + \frac{1}{3\pi} \left(1 - \frac{t^2 a_0^2}{a^2} \right) \sqrt{1 - \left(\frac{t a_0}{a} \right)^2}; \qquad (4.71)$$

$$k_z''' = -\frac{8}{3\pi} \left[\frac{3\varphi_z}{8} - \sin \frac{2\varphi_z}{4} + \sin \frac{4\varphi_z}{32} \right]. \qquad (4.72)$$
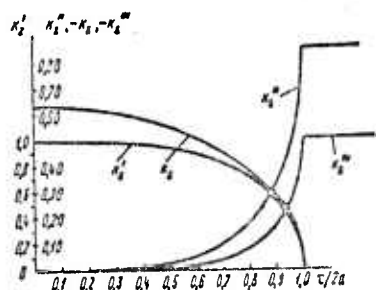
Fig. 93. Plot of the coefficients.

A plot of the coefficients k_z, k'_z, k''_z, and k'''_z is shown in Fig. 93.

The qualitative deductions obtained on the basis of an analysis of the hydrodynamic fields due to the translational displacement of a round disk can be extended to include also the case of motion of plates of other configurations in the liquid.

§7. ENTRY OF UNDERWATER SHOCK WAVE INTO A CHANNEL OF CONICAL CROSS SECTION

Closely related with diffraction problems is the question of the

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entry of an underwater shock wave into a pipe or a channel. Let us consider by way of an example the wave pattern occurring during the propagation of a shock wave in a channel of conical cross section. The walls of the channel will be assumed absolutely rigid. The solution will be developed in the acoustic approximation.*

As is well known, the potential of the direct wave, in the case of motion with spherical symmetry, can be written in the form

$$\varphi = -\frac{\psi(\tau - R)}{R} v_0 (\tau - R), \qquad (4.73)$$

where $\tau = t a_0 / R_{03}$ is the dimensionless time and $R = r / R_{03}$ is the dimensionless distance.

In order to make use of the boundary condition of the problem, according to which the value of the normal component of the liquid flow velocity should be equal to zero on the wall, let us calculate the value of this component on the surface of the cone. To this end we make use of a geometrical interpretation of the reflection process, the gist of which can be readily explained with the aid of Fig. 94.

Actually, if we construct a family of cones with vertex angles $(2i + 1)\alpha_0$ and consider a straight ray crossing the surfaces of these cones, then the lengths of the ranges of the reflected waves will be characterized by the quantities R_i and the zones of double and multiple reflections by the quantities L_i.

The values of R_i and L_i can be readily obtained from a direct examination of the reflection pattern. If $i < (\pi - 2\beta_0 - 2\alpha_0)/2\alpha_0$ (the ray R_{0i} crosses the generatrices of the cone), then

$$L_i = \frac{z_0 \sin \beta_0}{\sin [\beta_0 + (2i + 1)\alpha_0]}; \qquad (4.74)$$

$$R_{0i} = \sqrt{L_i^2 + z_0^2 - 2 z_0 L_i \cos (2i + 1)\alpha_0} =$$
$$= \frac{z_0 \sin (2i + 1)\alpha_0}{\sin [\beta_0 + (2i + 1)\alpha_0]}; \qquad (4.75)$$

where

$$\beta_0 = \text{arc tg} \frac{L_0 \sin \alpha_0}{z_0 - L_0 \cos \alpha_0}. \qquad (4.76)$$
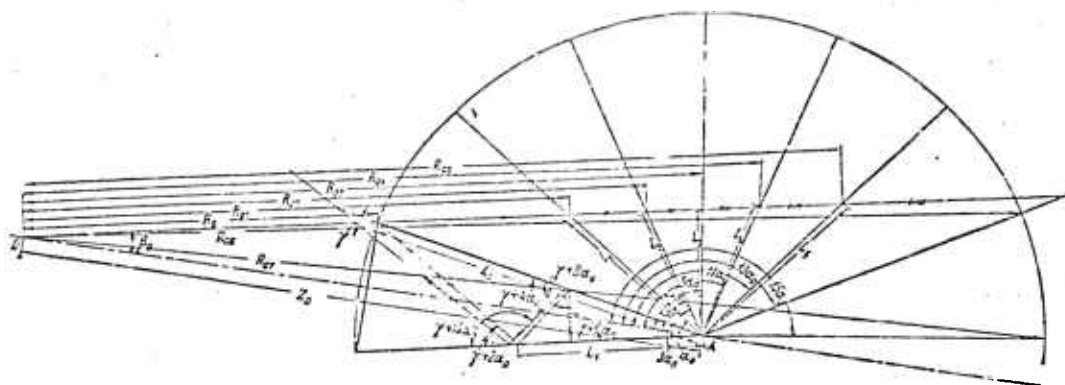
Fig. 94. Pattern showing the entry of a shock wave in an absolutely rigid cone.

When

$$l > \frac{\pi - 2\beta_0 - 2\alpha_0}{2\alpha_0}$$

we have

$$L_i = L_0; \qquad (4.77)$$

$$R_{0i} = \sqrt{L_0^2 + z_0^2 - 2z_0 L_0 \cos(r_i + 1)\alpha_0} \qquad (4.78)$$

(the ray R_{0i} emerges from the cone).

Returning to the potential (4.73), we obtain for the value of the normal velocity component on the surface of the cone

$$v_{ni} = \frac{\partial \varphi_i}{\partial n} = \frac{\partial \varphi_i}{\partial R_i} \frac{\partial R_i}{\partial n} =$$

$$= \frac{\partial R_i}{\partial n} \left\{ \frac{\psi(\tau - R_i)}{R_i^2} \sigma_0(\tau - R_i) - \right.$$

$$\left. - \frac{\frac{d}{dR_i} [\psi(\tau - R_i)\sigma_0(\tau - R_i)]}{R_i} \right\}. \qquad (4.79)$$

But the expression $\partial R_i / \partial n$ is the cosine of the angle between the direction of motion of the wave and the surface of the cone. For R_0, this angle is equal to $90 - \beta_0 - \alpha_0$ and changes by $2\alpha_0$ in each succeeding reflection.

Thus

$$\frac{\partial R_i}{\partial n} = \cos\{90 - \beta_0 - (2l + 1)\alpha_0\} = \sin\{(2l + 1)\alpha_0 + \beta_0\},$$

but according to (4.75)

$$\sin\left[\beta_0 + (2l+1)\,\alpha_0\right] = \frac{z_0}{R_{0l}}\sin\,(2l+1)\,\alpha_0,$$

consequently,

$$v_{n_l} = \frac{\partial \varphi_l}{\partial n_l} = \sin\,(2l+1)\,\alpha_0\,\frac{z_0}{R_l}\left\{\frac{\psi\,(\tau-R)}{R_l^2}\,\sigma_0\,(\tau-R) - \right.$$

$$\left. - \frac{\frac{d}{dR_l}\left[\psi\,(\tau-R_l)\,\sigma_0\,(\tau-R_l)\right]}{R_l}\right\}. \qquad (4.80)$$

Comparing (4.80) with (4.36) we note that the expression for the normal component of the velocity on the surface of the cone differs from that for the disk only in the presence of the factor $\sin(2i+1)\alpha_0$.

To estimate the potential at the vertex of the cone, we use the radiation integral, which in this case is written in the form

$$\varphi_A = \sum_{l=0}^{l=n}\int_0^{L_l}\sin\,(2l+1)\,\alpha_0\,\frac{z_0}{R_l}\left\{-\frac{\psi\,(\tau-R_l-l_l)}{R_l^2}\,\sigma_0\,(\tau-R_l-l_l) + \right.$$

$$\left. + \frac{\frac{d}{d\,(R_l+l_l)}\left[\psi\,(\tau-R_l-l_l)\,\sigma_0\,(\tau-R_l-l_l)\right]}{R_l}\right\}dl_l, \qquad (4.8\colon \qquad (4.81)$$

where R_i and l_i are continuously varying distances, determined by expressions analogous to (4.75)-(4.74):

$$l_l = \frac{z_0\sin\beta_0}{\sin\,[\beta_0 + \alpha]};$$

$$R_l = \frac{z_0\sin\alpha}{\sin\,[\beta_0 + \alpha]};$$

$$(2l+1)\,\alpha_0 < \alpha\,(2l+3)\,\alpha_0.$$

Calculation of this integral is carried out in the same way as in the cases previously considered. We therefore write here the final expression for the pressure at the vertex of the cone, with allowance for both the reflected and the direct waves:

$$p_A = -\rho_0\,\frac{1}{z_0}\,\frac{\partial}{\partial\tau}\,\psi\,(\tau-z_0)\,\sigma_0\,(\tau-z_0) - $$

$$- \sum_{l=0}^{l=n}\left\{\frac{\sin\,(2l+1)\,\alpha_0}{z_0\,[1-\cos\,(2l+1)\,\alpha_0]}\,\rho_0\,\frac{\partial}{\partial\tau}\,\psi\,(\tau-z_0)\,\sigma_0\,(\tau-z_0) - \right.$$

$$\left. - \rho_0\,\frac{z_0\sin\,(2l+1)\,\alpha_0}{R_{0l}\,[R_{0l}+L_l-z_0\cos\,(2l+1)\,\alpha_0]}\,\frac{\partial}{\partial\tau}\,\psi\,(\tau-R_{0l} - \right.$$

$$\left. - L_l)\,\sigma_0\,(\tau-R_{0l}-L_l)\right\}. \qquad (4.82)$$

In perfect analogy we can consider the problem of estimating the

potential at the top of a truncated cone. A geometric illustration of this process of reflection of the waves is shown in Fig. 95. Unlike the preceding case, some of the waves will emerge in this case from the truncated cone without participating in further reflection. Formally this fact can be taken into account in simplest fashion by characterizing the zones of multiple reflections not by the quantities L_i, but by the differences $L_i - l_{0i}$.

As before we have

$$L_i = \frac{z_0 \sin \beta_0}{\sin [\beta_0 + (2i + 1)\, a_0]},$$ (4.74)

if

$$i < \frac{\pi - 2\beta_0 - 2a_0}{2a_0}$$

and $L_i = L_0$, when $i > \frac{\pi - 2\beta_0 - 2a_0}{2a_0}$.
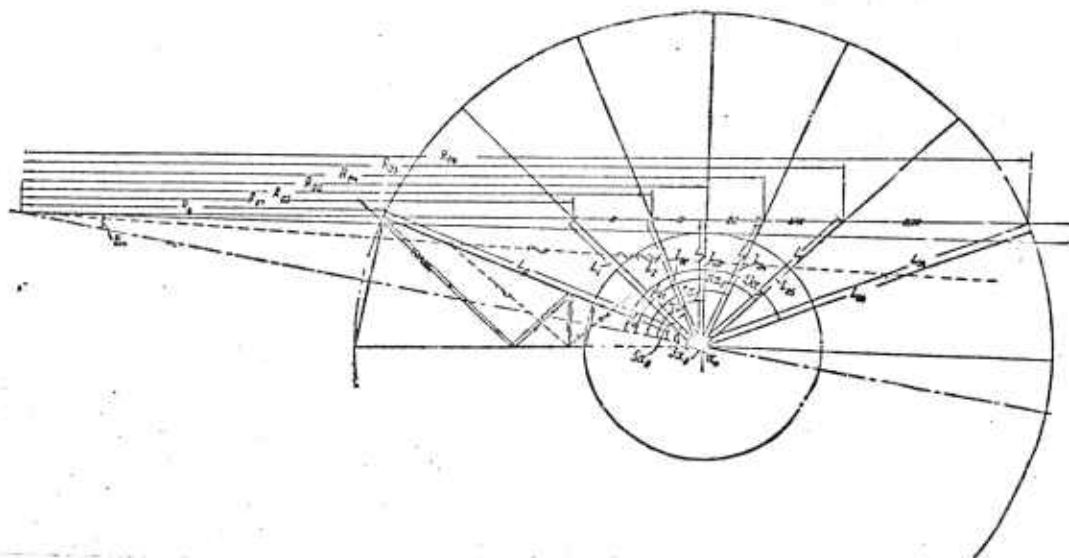
Fig. 95. Pattern of entry of shock wave in an absolutely rigid truncated cone.

As regards the segments l_{0i}, we have $l_{0i} = l_0$ so long as

$$\cos (2i + 1)\, a_0 \geqslant \frac{l_0}{z_0}$$

or, what is the same,

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$$l \leqslant \frac{\arccos \frac{l_0}{z_0} - a_e}{2a_0}.$$

As soon as

$$l > \frac{\arccos \frac{l_0}{z_0} - a_0}{2a_0},$$

we get

$$l_{ol} = \frac{l_0}{\cos \left[(2l + 1) a_0 - \arccos \frac{l_0}{z_0} \right]}. \tag{4.83}$$

Making use of the preceding scheme of solution, we can obtain the final result in the form

$$
\begin{aligned}
p_A = & -\rho_0 \frac{1}{z_0} \frac{\partial}{\partial \tau} \psi (\tau - z_0) \, \sigma_0 (\tau - z_0) - \\
& - \sum_{l=0}^{l=n} \Big\{ \frac{z_0 \sin (2l + 1) a_0}{r_{ol} [r_{ol} + l_{ol} - z_0 \cos (2l + 1) a_0]} \times \\
& \times \rho_0 \frac{\partial}{\partial \tau} \psi (\tau - r_{ol} - l_{ol}) \, \sigma_0 (\tau - r_{ol} - l_{ol}) - \\
& - \frac{z_0 \sin (2l + 1) a_0}{R_{ol} [R_{ol} + L_l - z_0 \cos (2l + 1) a_0]} \times \\
& \times \rho_0 \frac{\partial}{\partial \tau} \psi (\tau - R_{ol} - L_l) \, \sigma_0 (\tau - R_l - L_l) \Big\},
\end{aligned} \tag{4.84}
$$

where

$$r_{ol} = \sqrt{l_{ol}^2 + z_0^2 - 2z_0 l_{ol} \cos (2l + 1) a_0}. \tag{4.85}$$

Example. Calculate the pressure at the top of a truncated cone under the following initial data:

$$z_0 = 77.5 \text{ m}, \quad L_0 = 19.5 \text{ m}, \quad l_0 = 6.3 \text{ m},$$
$$a_0 = 0.301, \quad \beta_0 = 0.1, \quad G = 5000 \text{ kg}.$$

The explosive is TNT. Construct the pressure pattern.

Solution. To determine the coefficients of the functions characterizing the variation in pressure, we must calculate the following:

$$L_l = \frac{Z_0 \sin \beta_0}{\sin [\beta_0 + (2l + 1) a_0]}; \tag{4.74}$$

$$R_{0_l} = \sqrt{L_l^2 + Z_0^2 - 2Z_0 L_l \cos (2l + 1) a_0}; \tag{4.78}$$

$$l_{0_l} = \frac{l_0}{\cos \left[(2l + 1) a_0 - \arccos \frac{l_0}{Z_0} \right]}; \tag{4.83}$$

$$r_{0_l} = \sqrt{l_{0l}^2 + Z_0^2 - 2l_{0l}Z_0 \cos(2l+1)a_0};$$

$$k_1 = \frac{Z_0 \sin(2l+1)a_0}{r_{0l}[r_{0l} + l_{0l} - Z_0 \cos(2l+1)a_0]};$$

$$k_2 = \frac{Z_0 \sin(2l+1)a_0}{R_{0l}[R_{0l} + L_l - Z_0 \cos(2l+1)a_0]}.$$

TABLE 21

№ n/п.	1 Обозначения величины	$l=0$	$l=1$	$l=2$	$l=3$
1	$(2l+1)a_0$	18°30'	55°30'	92°30'	129°30'
2	$\cos(2l+1)a_0$	0,948	0,566	—0,044	—0,636
3	$\beta_0 + (2l+1)a_0$	24°30'	61°30'	98°30'	135°30'
4	$\sin[\beta_0 + (2l+1)a_0]$	0,415	0,879	0,989	0,700
5	$L_l = \dfrac{Z_0 \sin\beta_0}{\sin[\beta_0 + (2l+1)a_0]}$	19,0	9,20	8,15	11,5
6	$2Z_0 L_l \cos(2l+1)a_0$	2780	806	—57	—1135
7	$L_l^2 + Z_0^2$	6360	6085	6066	6132
8	(6) — (7)	3580	5279	6123	7267
9	$R_{0l} = \sqrt{(8)}$	59,7	72,5	78,4	85,2
10	$\cos\left[\arccos\dfrac{l_0}{Z_0} - (2l+1)a_0\right]$	0,383	0,870	0,992	0,717
11	$l_{0l} = \dfrac{l_0}{(10)}$	6,3	6,3	6,35	8,8
12	$2Z_0 l_{0l} \cos(2l+1)a_0$	926	552	—43	—866
13	$l_{0l} + Z_0^2$	6040	6040	6040	6078
14	(13) — (14)	5114	5188	6083	6944
15	$r_{0l} = \sqrt{(14)}$	71,5	74,0	77,9	83,3
16	$\sin(2l+1)a_0$	0,317	0,824	0,999	0,772
17	$Z_0 \sin(2l+1)a_0$	24,6	63,8	77,5	59,8
18	$Z_0 \cos(2l+1)a_0$	73,5	43,8	—3,4	—49,4
19	$r_{0l} + l_{0l}$	77,8	80,3	84,2	92,1
20	(19) — (18)	4,3	36,5	87,6	141,5
21	r_{0l} (20)	307	2800	6810	11 770
22	$k_1 = \dfrac{(17)}{(21)}$	0,080	0,023	0,0114	0,0051
23	$R_{0l} + L_l$	78,7	81,7	86,6	96,7
24	(23) — (18)	5,2	37,9	90,0	146,1
25	R_{0l} (24)	310	240	7050	12480
26	$k_2 = \dfrac{(17)}{(25)}$	0,079	0,023	0,0110	0,0048

1) Designation of quantity.

Calculation of the indicated quantities is summarized in Table 21. We obtain the relative distances characterizing the different reflection regions.

The radius of the equivalent spherical charge is equal to $R_{O3} =$
$= 0.053 \sqrt[3]{5000} = 90.6$ cm.

The quantities $r_{Oi} + l_{Oi}$ and $R_{Oi} + L_{Oi}$ are taken from Table 22.

TABLE 22

[1] Обозначения величин	[2] Абсолютные расстояния	[3] Относительные расстояния	[4] Интервалы времени между подходом волновых систем в вершину конуса
Z_0	77,5	85,5	0
$r_{00} + l_{00}$	77,8	85,9	0,4
$R_{00} + L_0$	78,7	86,8	1,3
$r_{01} + l_{01}$	80,3	88,6	3,1
$R_{01} + L_1$	81,7	90,1	4,6
$r_{02} + l_{02}$	84,2	92,9	7,4
$R_{02} + L_2$	86,6	95,5	10,0
$r_{03} + l_{03}$	92,1	103,5	18,0
$R_{03} + L_3$	96,7	114,0	28,5

1) Designations of the quantities; 2) absolute distances; 3) relative distances; 4) time intervals between arrival of wave systems at the top of the cone.

We measure time from the instant of arrival of the direct wave at the top of the cone.

We change over to dimensionless time and distances ($\overline{r} = r/R_{O3}$; $\tau = ta_0/R_{O3}$).

The pressure on the front of the direct wave at the top of the cone will be, in accord with (2.151),

$$p_m = 14700 \frac{1}{85,5^{1,13}} = 96,4 \text{ kg/cm}^2.$$

The change in pressure at this point, calculated by Formula (2.105), is summarized in Table 23.

Further calculations are elementary and are combined for simplicity with the graphical construction.

First to arrive at the top of the cone is the direct wave. The pressure in it varies in accordance with the data of Tables 22 and 23.

After a time interval equal to 0.4 (see Table 22), a compression

TABLE 23

τ	0	0,5	1,0	1,5	2,0	3,0
1 p, $\kappa z/c_{M^2}$	96,4	69,1	55,4	41,8	39,1	29,7
τ	5,0	7,0	10,0	15,0	20,0	40,0
1 p, $\kappa z/c_{M^2}$	18,6	12,0	6,9	2,7	0,86	0,39

1) p, kg/cm^2.

wave from the zone of the first reflection arrives at the point (it is characterized by coordinates $r_{00} + l_{00}$ and by a coefficient k_{10}). The pressure at the point is made up of the pressures in the direct and reflected waves, the amplitude of the latter also varying in accordance with the law given by Table 23, with a coefficient $k_1/1/z_0 =$

$= 0.080 \cdot 77.5 = 6.2$.
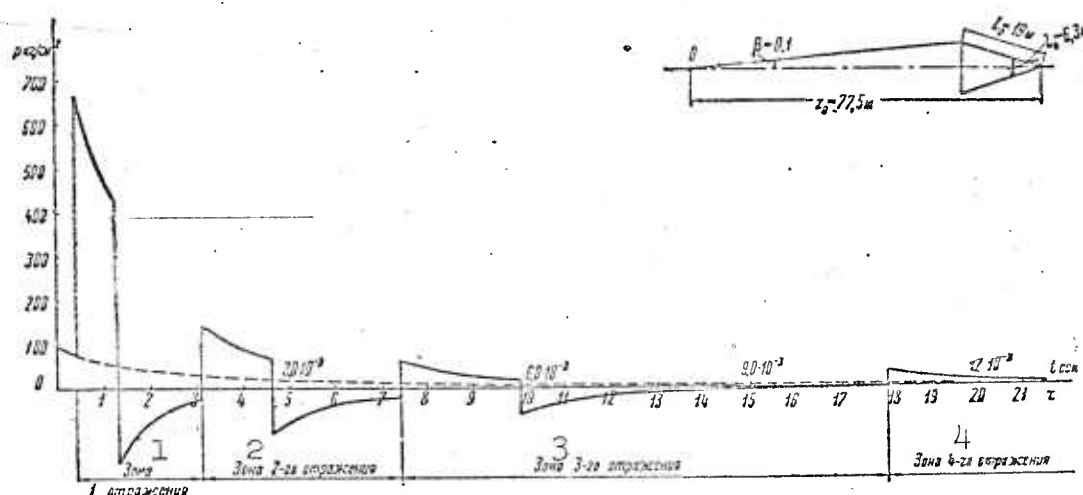
Fig. 96. Calculated pressure pattern at the top of a truncated cone. Solid line — pressure at the point with allowance for the entry of the shock wave into the cone; dashed — pressure at a point of the free liquid. 1) First reflection zone; 2) second reflection zone; 3) third reflection zone; 4) fourth reflection zone.

After a lapse $\tau = 1.3$ from the initial instant, a rarefaction wave (second term of Formula (4.84), characterized by the coordinates R_{00} and L_0 and by the coefficient k_2) arrives at the top of the cone.

The pressure will be determined by the superposition of the three wave systems.

Further development of the process is perfectly similar to that described above. The results of the calculations are presented in Fig. 96.

It is easy to see that as a result of the multiple reflections, the pressure produced when a shock wave enters into a cone may turn out to be much higher than in the free liquid.

§8. GENERAL CONSIDERATIONS IN THE DYNAMIC DESIGN OF STRUCTURES TO WITHSTAND THE ACTION OF AN UNDERWATER SHOCK WAVE

The strength of any particular structure, designed to withstand the action of an explosion, is determined in final analysis by the propagation of the elastoplastic waves arising in the structure as a result of reflection and refraction of the shock wave. This process therefore depends on the structure itself. We shall consider here the simplest structure, made up of plates.

According to the general theory, when a shock wave is incident on a plate, there arise in addition to the direct and reflected waves also three other types of waves: longitudinal, transverse, and surface. Owing to multiple reflection and interaction, the over-all wave pattern becomes so complicated, even in the study of the simplest cases, that the known methods of mathematical analysis are incapable of answering the question of strength estimate when so formulated. It is therefore necessary to introduce many simplifying assumptions in the solution of practical problems.

As a rule, the structures are regarded as mechanical systems with one or several degrees of freedom. The wave character of the process is disregarded. The external load is replaced by a generalized force acting in accordance with some definite law.

We shall show in what follows that in many cases such an approach to the solution of the problem is permissible.

At the same time, the following factors must be taken into account.

1. The magnitude and character of the external load, perceived by the structure, depends not only on the pressure fields produced by the explosion in the free liquid, but also to a considerable degree on the elastoplastic characteristics of the structure itself. The deformation of the structure due to the action of the shock wave leads in turn to a change in the pressure fields. The considerable acoustic impedance of water makes this mutual influence most essential.

2. Under certain conditions the interaction between the shock wave and the deforming partition in the liquid can give rise to cavitation regions, the influence of which calls for appreciable correction of the estimate of the external forces.

3. The pressure fields are influenced in a complicated manner by the free surface of the liquid. As was mentioned in §§2 and 3 of Chapter 3, the presence of the free surface leads to a change in all the parameters of the underwater shock wave: pressure on the front, duration of the positive phase, and form of the pressure-time pattern.

4. Even more complicated, as is obvious from the materials of the preceding chapter, is the estimate of the pressure fields when the bottom of the reservoir exerts a noticeable influence. In addition to everything else, it is necessary to take into consideration the difference in the times that the individual wave systems arrive at the structure.

5. In most cases the final judgment regarding the strength of one structure or another under the influence of an explosion can be made by considering not only the elastic but also the plastic deformations. It is frequently necessary to take into consideration the strengthening of the material under large strain rates. At the present time, even

static calculations in the plasticity zone entail serious difficulties and cannot be regarded as a question that has been given due study. The difficulties presented by the problem under consideration are therefore clear.

There is an extensive literature devoted to the action of noncontact underwater explosions on structures. Even a brief exposition of the available material would call for a separate book. We therefore develop below only some particular problems in this field, without any pretense whatever to completeness, aimed merely at giving a general idea of the subject considered.

§9. METHOD OF GENERALIZED COORDINATES AS APPLIED TO THE PROBLEM OF DYNAMIC CALCULATION OF STRENGTHS OF STRUCTURES UNDER NONCONTACT UNDERWATER EXPLOSIONS

To solve the dynamic problems under consideration, it is advantageous to use the Lagrange equations of the second kind, usually written in the form

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} = Q_j, \tag{4.86}$$

where T is the kinetic energy of the system, q_j is the generalized coordinate, and Q_j the generalized force corresponding to the coordinate q_j.

To determine the value of the generalized force Q_j one usually employs one of the two following methods:

1) the generalized force is determined from the coefficient of the generalized displacement δq_j in the expression for the elementary work performed by the specified forces in the aggregate of the possible displacements of the system, or

2) the value of Q_j is determined from the partial derivative of the potential energy of the system with respect to the generalized coordinate:

$$Q_j = -\frac{\partial \Pi}{\partial q_j}. \tag{4.87}$$

The Lagrange method as applied to dynamic design of ship structures was developed by Yu. A. Shimanskiy.

Considering systems consisting of elastic bodies, and noting that in this case the number of degrees of freedom is generally infinite, Shimanskiy shows that if the form of the oscillations is known the problem reduces to a study of a system with a finite number of degrees of freedom, since an oscillation of specified form is determined completely by some single generalized coordinate, for example the displacement of a chosen point of the body. Such a point is called the reduction point by Shimanskiy.

The displacement of the reduction point can be determined by choosing the form of the oscillations (the form of the deflection) and determining the so-called reduced mass, the reduced stiffness coefficient, and the reduced force.

Let us illustrate this by means of an example.

Let $f(x, y, z)$ be a function of the coordinates of the points of the body, determining the form of the oscillation of the body under consideration. For the reduction point we shall assume that $f = 1$; let $m(x, y, z)$ be a function of the distribution of the mass of the body; $F(x, y, z)$ a function of the distribution and of the intensity of the external load; and $\varphi = \varphi(t)$ the equation of motion of the reduction point.

The equation of motion of an arbitrary point of the body will obviously be $\varphi = \varphi(t) f(x, y, z)$.

Let us calculate the reduced mass.

The kinetic energy of motion of the body is determined by the equation

$$T = \frac{1}{2} \iiint m \varphi'^2(t) f^2(x, y, z)\, dx\, dy\, dz =$$
$$= \frac{1}{2} \varphi'(t) \iiint m f^2\, dx\, dy\, dz. \tag{4.88}$$

In order for the reduction point to have the same kinetic energy it must obviously have a reduced mass

$$M_{np} = \iiint m f^2 dx dy dz. \qquad (4.89)$$

Consequently, the reduced mass is equal to the sum of the products of the masses of all the points of the body by the squares of their displacements, assuming the displacement of the reduction point equal to unity.

As is well known, the stiffness coefficient $\underline{k}$ is called the coefficient of proportionality between the strain (the displacement φ) and the value of the force $\underline{p}$:

$$p = k\varphi. \qquad (4.90)$$

Since the potential energy of the elastic couplings is

$$V = \frac{p\varphi}{2} = \frac{k\varphi^2}{2}, \qquad (4.91)$$

we have

$$k = 2V/\varphi^2. \qquad (4.92)$$

Thus, the reduced stiffness coefficient is equal to twice the value of the potential energy of the body subjected to a displacement corresponding to unity displacement of the reduction point.

Let us determine now the reduced force.

It is obvious that when the reduction point is displaced by an amount $\Delta\varphi$, the other points of the system are displaced by amounts $\Delta\varphi f$ and the external forces applied to them perform work

$$\iiint \Delta\varphi(t) f(x, y, z) F(x, y, z) dx dy dz = $$
$$= \Delta\varphi \iiint fF \, dx dy dz. \qquad (4.93)$$

Equating this work to the work of the reduced force applied to the reduction point and equal to $\Phi_{pr}\Delta\varphi$, we obtain

$$\Phi_{np} = \iiint Ff \, dx dy dz. \qquad (4.94)$$

Consequently, the reduced force is equal to the products of the

displacements of the point of the body corresponding to unity displacement of the reduction point by the forces applied to these points.

The calculation of the potential and kinetic energy of different elastic systems does not entail any serious difficulties. This question was considered in sufficient detail for various dynamic problems in the already mentioned work by Yu. A. Shimanskiy "Dynamic Design of Ship Structures."

In the analysis of the action of an underwater explosion on a structure, it is only the generalized forces that are estimated differently. In addition to the forces of resistance to the deformation, characterized by the reduced stiffness coefficient, it is necessary to determine the hydrodynamic forces in this case also. These can be represented by the following:

1) the forces acting in an underwater explosion on the structure regarded as an absolutely rigid partition of finite dimensions;

2) the forces resulting from the deformation and displacement of the partition itself.

Previously we have already become acquainted, by means of simple examples, with methods of estimating the hydrodynamic forces of both the first and second category.

It should be pointed out that the need to account for the influence exerted on the hydrodynamic fields by the deformations and the displacement of the partition itself leads to a solution of the integral-differential equations.

Indeed, let us calculate the reduced mass, the reduced stiffness coefficient, and the generalized hydrodynamic force of the first category, Q_1, for the incidence of a plane wave on a round plate. Then, in accord with (4.64), the equation characterizing the deformation of the plate at the initial period of motion under the influence of an under-

water shock wave is written in the form

$$M\ddot{z}(t) + F(z, \dot{z}) = Q_1 + 2a^2\rho_0 a_0 \left\{ \frac{\pi}{2} \dot{z}(t) \sigma_0(t) - \frac{a_0}{a} z(t) \div \right.$$

$$\left. + \frac{a_0}{a} \int_0^{2\pi} z\left(t - \frac{2a}{a_0} \sin \psi\right) \sigma_0\left(t - \frac{2a}{a_0} \sin \psi\right) \sin \psi d\psi \right\} \qquad (4.95)$$

The sought function is under the integral sign in the right half of the equation.

Serious mathematical difficulties arise, connected with the solution of the integral-differential equation (4.95), as a result one usually makes from the very outset supplementary simplifying assumptions, with the aid of which it is possible to get rid of difficulties of this type. We shall dwell on this problem briefly later on. At present we proceed to an estimate of the hydrodynamic forces of the first category in the simplest cases of horizontal and vertical partitions.

§10. GENERALIZED HYDRODYNAMIC FORCES ON AN ABSOLUTELY RIGID VERTICAL
OR HORIZONTAL PARTITION IN AN UNDERWATER EXPLOSION

In underwater explosions near the free surface, the main elements of the shock wave, apart from the relative distance depend on the depth of the charge H and on the depth of the measurement point $\underline{h}$:

$$\left. \begin{array}{l} p_m = p_m(r, H, h), \\ t_+ = t_+(r, H, h). \end{array} \right\} \qquad (4.96)$$

The form of the pressure vs. time curve also changes appreciably.

Considering a region that is remote from the center of the explosion and regarding the distance $\underline{r}$ and the depth H of the charge as specified, we can expand the functions p_m and t_+ in a Taylor series about $h = 0$. We then obtain

$$p_m(r, H, h) = p_m(r, H, 0) + h \left.\frac{\partial p_m}{\partial h}\right|_{h=0} + \frac{1}{2} h^2 \left.\frac{\partial^2 p_m}{\partial h^2}\right|_{h=0} + \ldots \qquad (4.97)$$

$$t_+(r, H, h) = t_+(r, H, 0) + h \left.\frac{\partial \tau_+}{\partial h}\right|_{h=0} + \frac{1}{2} h^2 \left.\frac{\partial^2 \tau_+}{\partial h^2}\right|_{h=0} + \ldots \qquad (4.98)$$

With good approximation we can assume in the majority of cases of

practical interest that if the range of variation of the depths of the measurement points is given by the interval $0 < h < h_1$, then the average value of the pressure in this range of depths amounts to $p_{m_{sr}} = p_m(r, H, h_1/2)$.

As regards the duration of the compression phase, since $t_+ = 0$ on the free surface we obtain on the basis of (4.98), discarding terms of higher order of smallness,
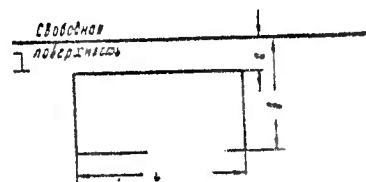
$$t_+ = Ah, \qquad (4.99)$$

where

$$A = \frac{\partial t_+}{\partial h}. \qquad (4.100)$$

Fig. 97. Diagram showing location of plate for an estimate of the resultant load in the case of normal incidence of the wave. 1) Free surface.

The simplest and most convenient form of analytic approximation of the pressure vs. time curve near the free surface is

$$p = p_m\left[1 - \left(\frac{t}{t_+}\right)^n\right], \qquad (4.101)$$

where p_1 is the pressure on the front, t_+ is the time of action of the positive phase of the pressure, and $\underline{t}$ is the running time, reckoned from the instant of arrival of the shock wave at the specified point $(t \leq t_+)$.

The exponent $\underline{n}$ is in the general case a function of $\underline{r}$, H, and $\underline{h}$. However, for specified $\underline{r}$ and H we can approximately assume for the coefficient $\underline{n}$ a constant value calculated for the center point of the plate.

In accordance with the foregoing, we find that

$$p = p_m\left[1 - \left(\frac{t}{Ah}\right)^n\right][\sigma_0(t) - \sigma_0(t - Ah)]. \qquad (4.102)$$

Let us find the resultant force acting in normal incidence of a wave on an absolutely rigid vertical plate immersed in a liquid as shown in Fig. 97, under the assumption that the pressure field is spec-

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ified by means of Eq. (4.102).*

We assume that the coefficient of reflection from the partition is equal to κ. We then get

$$Q_i = \int_s (1 + \varkappa)\, p\, ds =$$
$$= l p_m (1 + \varkappa) \int_a^b \left[1 - \left(\frac{t}{Ah} \right)^n \right] [\sigma_0(t) - \sigma_0(t - Ah)]\, dh. \qquad (4.103)$$

After simple transformations we have

$$Q_1 = (1 + \varkappa)\, l p_m \left\{ b \left[1 + \frac{1}{n-1} \left(\frac{t}{Ab} \right)^n \right] [\sigma_0(t) - \sigma_0(t - Ab)] - \right.$$
$$- a \left[1 + \frac{1}{n-1} \left(\frac{t}{Aa} \right)^n \right] [\sigma_0(t) - \sigma_0[(t - Aa)] +$$
$$\left. + \frac{t}{A} \frac{n}{n-1} [\sigma_0(t - Ab) - \sigma_0(t - Aa)] \right\}. \qquad (4.104)$$

Formula (4.104) is valid for all $\underline{n}$, including n = 1.
In this latter case

$$Q_1 = (1 + \varkappa)\, l p_m \left\{ \left[b - \frac{t}{A} \ln \frac{Ab}{t} \right] [\sigma_0(t) - \sigma_0(t - Ab)] - \right.$$
$$- \left[a - \frac{t}{A} \ln \frac{Aa}{t} \right] [\sigma_0(t) - \sigma_0(t - Aa)] +$$
$$\left. + \frac{t}{A} [\sigma_0(t - Ab) - \sigma_0(t - Aa)] \right\}. \qquad (4.105)$$

An estimate of the load on the horizontal plate differs somewhat from the case considered here.** This difference is due, first of all, to the lack of a reflected wave and, second, to the fact that as a result of the gliding of the wave along the partition the load first increases gradually and then, if the partition is longer than the wavelength, it becomes constant and then decreases (Fig. 98).

It is important to note that in the second case, as can be seen from Fig. 98, the value of the constant load does not depend on the dimensions of the partition, and is determined only by the length λ of the shock wave.

The value of the pressure at an arbitrary point of the plate with coordinate l is determined by the relation

$$p = p_m \left[1 - \left(\frac{t - \frac{l}{a_0}}{t_+} \right)^n \right] \left[\sigma_0 \left(t - \frac{l}{a_0} \right) - \sigma_0 \left(t - t_+ - \frac{l}{a_0} \right) \right]. \qquad (4.106)$$

Integrating (4.106) over the area we obtain after simple transformations the component of the general force, dependent on the length of the plate l_1

$$\overline{Q} = p_m s \left\{ \left[\frac{a_0 t}{l_1} - \frac{\lambda}{(n+1)\, l_1} \left(\frac{t}{t_+} \right)^{n+1} \right] [\sigma_0(t) - \sigma_0(t - t_+)] + \right.$$

$$\left. + \left[1 - \frac{a_0 t}{l_1} + \frac{\lambda}{(n+1) l_1} \left(\frac{t - \frac{l_1}{a_0}}{t_+} \right)^{n+1} \right] \left[\sigma_0 \left(t - \frac{l_1}{a_0} \right) - \sigma_0 \left(t - t_+ - \frac{l_1}{a_0} \right) \right] \right\}. \qquad (4.107)$$

It is necessary to add to this component, which takes into account the nonstationary nature of the wave advancing on the plate, a constant component which can be readily obtained by integrating the load with respect to the wavelength

$$\overline{Q} = p_m m \int_0^\lambda \left[1 - \left(\frac{l_1}{\lambda} \right)^n \right] dl = p_m\, m \left\{ l - \frac{\lambda}{n+1} \left(\frac{l}{\lambda} \right)^{n+1} \right\} \Big|_0^\lambda =$$

$$= p_m m \left\{ \lambda - \frac{\lambda}{n+1} \right\} = p_m s \frac{n}{n+1} \frac{\lambda}{l_1}, \qquad (4.108)$$

where

$$t_+ < t < t_+ + \frac{l_1}{a_0}.$$
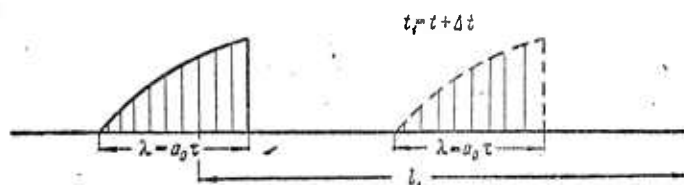
Fig. 98. Diagram showing the gliding of a shock wave along a horizontal plate.

Thus, we obtain ultimately

$$Q = p_m s \left\{ \left[\frac{a_0 t}{l_1} - \frac{\lambda}{(n+1)\, l_1} \left(\frac{t}{t_+} \right)^{n+1} \right] [\sigma(t) - \sigma_0(t - t_+)] + \right.$$

$$+ \frac{n}{n+1} \frac{\lambda}{l_1} \left[\sigma(t - t_+) - \sigma_0 \left(t - t_+ - \frac{l_1}{a_0} \right) \right] + \left[1 - \frac{a_0 t}{l_1} - \right.$$

$$\left. - \frac{\lambda}{(n+1) l_1} \left(\frac{t - \frac{l_1}{a_0}}{t_+} \right)^{n+1} \right] \left[\sigma_0 \left(t - \frac{l_1}{a_0} \right) - \sigma_0 \left(t - t_+ - \frac{l_1}{a_0} \right) \right] \right\}. \qquad (4.109)$$

Calculation performed with the aid of Formulas (4.102)-(4.109)

enable us to draw the following conclusions.

1. The generalized force on a vertical rigid partition can be calculated approximately on the basis of the plane-wave hypothesis. The pressure pattern must in this case be assumed for the center point of the plate.

2. When estimating the generalized force on a horizontal partition, it is necessary to take into account the gradual advance of the wave on the partition.

The character of the time variation of the generalized force differs essentially from the pressure pattern in a free liquid.

Example 1. Calculate the generalized force produced on the vertical absolutely rigid partition upon incidence of waves with triangular and parabolic profiles (n = 1 and n = 2) for the following initial data:

submerged depth of upper edge of plate a = 2.1 m,

submerged depth of lower edge of plate b = 6.3 m,

coefficient of proportionality between the time of action of the positive pressure phase, t_+, and the depth of the measurement point, h, is assumed to be A = 0.2·10^{-3} sec/m.

Compare the generalized force with the change in load at the center point of the plate.

Solution. The calculation of the generalized force for exponents n = 2 and n = 1 is carried out with the aid of the formulas

$$Q_1 = (1 + \varkappa)\, l p_m \left\{ b \left[1 + \frac{1}{n-1} \left(\frac{t}{Ab} \right)^n \right] [\sigma_0(t) - \sigma_0(t - Ab)] - \right.$$
$$- a \left[1 + \frac{1}{n-1} \left(\frac{t}{Aa} \right)^n \right] [\sigma_0(t) - \sigma_0(t - Aa)] +$$
$$+ \left. \frac{t}{A} \frac{n}{n-1} [\sigma_0(t - Ab) - \sigma_0(t - Aa)] \right\}; \qquad (4.104)$$

$$Q_1 = (1 + \varkappa)\, l p_m \left\{ \left[b - \frac{t}{A} \ln \frac{Ab}{t} \right] [\sigma_0(t) - \sigma_0(t - Ab)] - \right.$$
$$- \left[a - \frac{t}{a} \ln \frac{Aa}{t} \right] [\sigma_0(t) - \sigma_0(t - Aa)] +$$
$$+ \left. \frac{t}{A} [\sigma_0(t - Ab) - \sigma_0(t - Aa)] \right\}. \qquad (4.105)$$

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TABLE 24

$t \cdot 10^{-3}$, sec	$\dfrac{1}{b-a}$	Ab	$\dfrac{t}{Ab}$	$(4)^2$	$1+(5)$	$b \cdot (6)$	Aa
1	2	3	4	5	6	7	8
0,1	0,238	$1,26 \cdot 10^{-3}$	0,0794	0,00632	1,006	6,338	$0,42 \cdot 10^{-3}$
0,2	0,238	$1,26 \cdot 10^{-3}$	0,159	0,0253	1,025	6,45	$0,42 \cdot 10^{-3}$
0,3	0,238	$1,26 \cdot 10^{-3}$	0,238	0,0566	1,057	6,66	$0,42 \cdot 10^{-3}$
0,4	0,238	$1,26 \cdot 10^{-3}$	0,317	0,100	1,100	6,93	$0,42 \cdot 10^{-3}$
0,5	0,238	$1,26 \cdot 10^{-3}$	0,397	0,158	1,158	7,29	$0,42 \cdot 10^{-3}$
0,6	0,238	$1,26 \cdot 10^{-3}$	0,476	0,227	1,227	7,73	$0,42 \cdot 10^{-3}$
0,7	0,238	$1,26 \cdot 10^{-3}$	0,556	0,309	1,309	8,24	$0,42 \cdot 10^{-3}$
0,8	0,238	$1,26 \cdot 10^{-3}$	0,635	0,403	1,403	8,82	$0,42 \cdot 10^{-3}$
0,9	0,238	$1,26 \cdot 10^{-3}$	0,714	0,510	1,510	9,51	$0,42 \cdot 10^{-3}$
1,0	0,238	$1,26 \cdot 10^{-3}$	0,791	0,630	1,630	10,28	$0,42 \cdot 10^{-3}$
1,1	0,238	$1,26 \cdot 10^{-3}$	0,873	0,762	1,762	11,10	$0,42 \cdot 10^{-3}$
1,2	0,238	$1,26 \cdot 10^{-3}$	0,951	0,912	1,912	12,10	$0,42 \cdot 10^{-3}$
1,3	0,238	$1,26 \cdot 10^{-3}$	1,03	1,061	2,06	13,00	$0,42 \cdot 10^{-3}$

TABLE 25

$t \cdot 10^{-3}$, sec	$\dfrac{1}{b-a}$	$\dfrac{t}{A}$	$\dfrac{Ab}{t}$	$\ln \dfrac{Ab}{t}$	$\dfrac{t}{A}(5)$	$b-(6)$
1	2	3	4	5	6	7
0,1	0,238	0,5	12,6	2,534	1,265	5,035
0,2	0,238	1,0	6,9	1,841	1,841	4,459
0,3	0,238	1,5	4,2	1,435	2,150	4,15
0,4	0,238	2,0	3,15	1,147	2,296	4,00
0,5	0,238	2,5	2,52	0,924	2,310	3,99
0,6	0,238	3,0	2,10	0,742	2,220	4,08
0,7	0,238	3,5	1,80	0,588	2,060	4,24
0,8	0,238	4,0	1,58	0,454	1,816	4,48
0,9	0,238	4,5	1,40	0,337	1,510	4,79
1,0	0,238	5,0	1,26	0,231	1,155	5,15
1,1	0,238	5,5	1,15	0,135	0,744	5,56
1,2	0,238	6,0	1,03	0,049	0,294	6,01
1,3	0,238	6,5	0,97	−0,0305	−0,198	6,50

Choosing as the time interval $\Delta t = 0.1 \cdot 10^{-3}$ sec, we carry out the calculations in tabular form (Tables 24 and 25).

The results of the calculations are presented in Figs. 99 and 100. The same figures show (dashed) the change in pressure at the center of the plate ($h = (a + b)/2 = 4.2$ m), obtained with the aid of Relation (4.102) (Table 26).

Example 2. Calculate the general force arising when a shock wave glides against a horizontal plate under the following initial condi-

$\dfrac{t}{Aa}$	$(9)^2$	$1+(10)$	$a\,(11)$	$(7)-(12)$	$\dfrac{t}{A}\,2$	$(7)-(14)$	$Q_1=$ $=(2)\cdot(15)$	$Q_1=$ $=(2)\cdot(13)$
9	10	11	12	13	14	15	16	17
0,238	0,0566	1,056	2,217	4,121	—	—	—	0,960
0,476	0,226	1,226	2,580	3,87	—	—	—	0,920
0,714	0,510	1,510	3,170	3,49	—	—	—	0,830
0,954	0,908	1,908	4,000	2,93	—	—	—	0,698
—	—	—	—	—	5,0	2,29	0,546	—
—	—	—	—	—	6,0	1,73	0,412	—
—	—	—	—	—	7,0	1,24	0,295	—
—	—	—	—	—	8,0	2,82	0,195	—
—	—	—	—	—	9,0	0,51	0,122	—
—	—	—	—	—	10,0	0,28	0,067	—
—	—	—	—	—	11,0	0,10	0,024	—
—	—	—	—	—	12,0	0,03	0,012	—
—	—	—	—	—	13,0	0,00	0,000	—

$\dfrac{Aa}{t}$	$\ln\dfrac{Aa}{t}$	$\dfrac{t}{A}\,(9)$	$a-(10)$	$(7)-(11)$	$Q_1=$ $=(2)\cdot(12)$	$(7)-(3)$	$Q_1=$ $=(2)\cdot(14)$
8	9	10	11	12	13	14	15
4,2	1,435	0,717	1,383	3,652	0,868	—	—
2,1	0,742	0,742	1,358	3,100	0,736	—	—
1,4	0,337	0,514	1,586	2,564	0,608	—	—
1,05	0,049	0,098	2,002	2,00	0,476	—	—
—	—	—	—	—	—	1,450	0,345
—	—	—	—	—	—	1,080	0,257
—	—	—	—	—	—	0,740	0,176
—	—	—	—	—	—	0,480	0,114
—	—	—	—	—	—	0,290	0,0698
—	—	—	—	—	—	0,145	0,0345
—	—	—	—	—	—	0,056	0,0133
—	—	—	—	—	—	0,005	0,00119
—	—	—	—	—	—	0,00	0,00

tions:

time of action of the compression phase $t_+ = 1.27 \cdot 10^{-3}$ sec,

wave length $\lambda = 1.9$ m,

length of plate $l_1 = 4.2$ m.

The exponent n is assumed equal to zero.

Solution. Calculation of the generalized force is by means of

Formula (4.109):
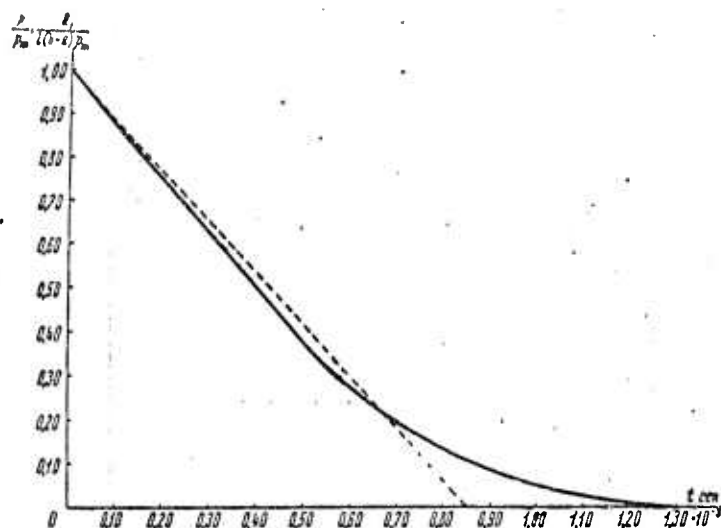
Fig. 99. Plot of generalized hydrodynamic force on a rigid vertical partition for n = 1.

$$-\frac{Q_1}{l\,(b-a)\,P_m} = \left[\frac{b}{b-a} - \frac{t}{A\,(b-a)}\ln\frac{Ab}{t}\right][\sigma_0(t) - \sigma_0(t-Ab)] -$$

$$-\left[\frac{a}{b-a} - \frac{t}{A\,(b-a)}\ln\frac{Aa}{t}\right][\sigma_0(t) - \sigma_0(t-Aa)] +$$

$$+\frac{t}{A\,(b-a)}[\sigma_0(t-Ab) - \sigma_0(t-Aa)];$$

$$----\frac{p}{p_m} = 1 - \frac{t}{A h}.$$
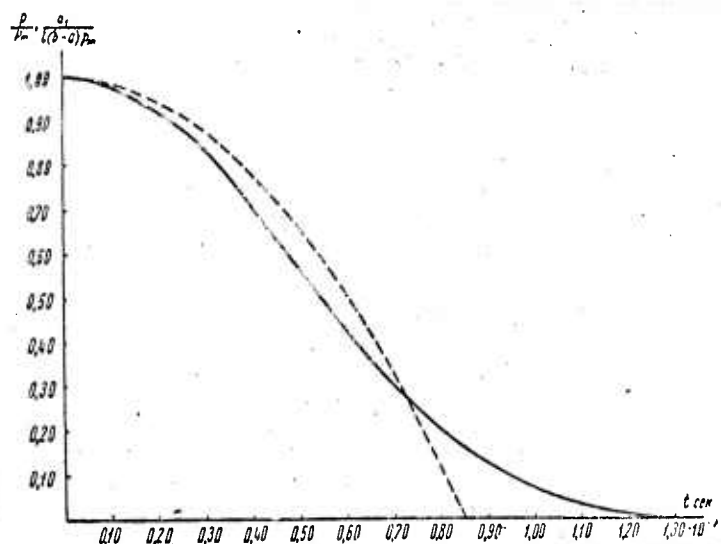
Fig. 100. Same as Fig. 99 for n = 2.

$$-\frac{Q_1}{l\,(b-a)\,P_m} = \frac{b}{b-a}\left[1 + \frac{1}{a-1}\left(\frac{t}{Ab}\right)^a\right][\sigma_0(t) - \sigma_0(t-Ab)] -$$

$$-\frac{a}{b-a}\left[1 + \frac{1}{n+1}\left(\frac{t}{Aa}\right)^a\right][\sigma_0(t) - \sigma_0(t-Aa)] +$$

$$+\frac{t}{A\,(b-a)}\frac{n}{n-1}[\sigma_0(t-Ab) - \sigma_0(t-Aa)];$$

$$----\frac{p}{p_m} = 1 - \left(\frac{t}{Ah}\right)^a.$$

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$$Q = p_m S\left\{\left[\frac{a_0 t}{l_1} - \frac{\lambda}{(n+1)l_1}\left(\frac{t}{t_1}\right)^{n+1}\right][\sigma_0(t) - \sigma_0(t-t_1)] +\right.$$

$$+ \frac{n}{n+1}\frac{\lambda}{l_1}\left[\sigma_0(t-t_+) - \sigma_0\left(t-t_+ - \frac{t_1}{a_0}\right)\right] +$$

$$+ \left|1 - \frac{a_0 t}{l_1} - \frac{\lambda}{(n+1)l_1}\cdot\left(\frac{t - \frac{l_1}{a_0}}{t_+}\right)^{n+1}\right| \times$$

$$\left. \times \left[\sigma_0\left(t - \frac{l_1}{a_0}\right) - \sigma_0\left(t - t_+ - \frac{l_1}{a_0}\right)\right]\right\}.$$

TABLE 26

$t\cdot 10^{-3}$, sec	Ah	$\frac{t}{Ah}$	$\left(\frac{t}{Ah}\right)^2$	$\dfrac{p}{p_m}$	
				$n = 2$	$n = 1$
0,1	0,84	0,119	0,0143	0,986	0,881
0,2	0,84	0,238	0,0566	0,944	0,762
0,3	0,84	0,357	0,127	0,873	0,643
0,4	0,84	0,476	0,226	0,774	0,524
0,5	0,84	0,596	0,355	0,645	0,404
0,6	0,84	0,715	0,511	0,489	0,285
0,7	0,84	0,834	0,696	0,305	0,166
0,8	0,84	0,952	0,906	0,094	0,048
0,84	0,84	1,000	1,000	0,000	0,000

Choosing a time interval $\Delta t = 0.2\cdot 10^{-3}$ sec, we carry out the calculations in tabular form (Table 27).

The results of the calculations are presented on Fig. 101.
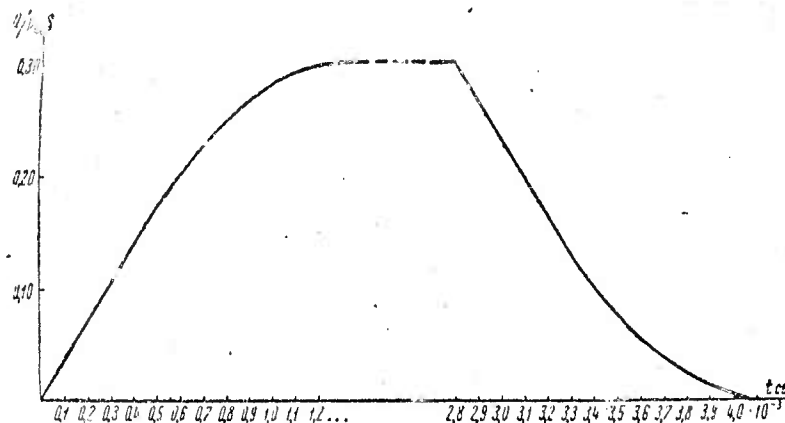
Fig. 101. Plot of generalized hydrodynamic force on a rigid horizontal partition for $n = 2$

$$\frac{Q}{p_m S} = \left[\frac{a_0 t}{l_1} - \frac{\lambda}{(n+1)l_1}\left(\frac{t}{\tau}\right)^{n+1}\right][\sigma_0(t) - \sigma_0(t-\tau)] +$$

$$+ \left[1 - \frac{a_0 t}{l_1} - \frac{\lambda}{(n+1)l_1}\left(\frac{t - \frac{l_1}{a_0}}{\tau}\right)^{n+1}\right]\left[\sigma_0\left(t - \frac{l_1}{a_0}\right) - \sigma_0\left(t - \tau - \frac{l_1}{a_0}\right)\right] +$$

$$+ \frac{\lambda}{l_1}\frac{n}{n+1}\left[\sigma_0(t-\tau) - \sigma_0\left(t - \tau - \frac{l_1}{a_0}\right)\right].$$

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TABLE 27

$t \cdot 10^{-2}$, sec	$\dfrac{t}{1,27 \cdot 10^{-3}}$	$(2)^2$	$0,151 \cdot (3)$	$\dfrac{a_0 t}{l_1}$	$Q = (5)-(4)$	$\dfrac{l_1}{a_0}$	$t-(7)$
1	2	3	4	5	6	7	8
0,2	0,158	0,00387	0,000584	0,0714	0,0708	—	—
0,4	0,315	0,0313	0,00173	0,143	0,135	—	—
0,6	0,473	0,106	0,0160	0,214	0,198	—	—
0,8	0,630	0,250	0,0378	0,286	0,249	—	—
1,0	0,797	0,506	0,0764	0,357	0,281	—	—
1,2	0,945	0,844	0,127	0,428	0,301	0,00288	0,00020
3,0	—	—	—	—	—	0,00288	0,00040
3,2	—	—	—	—	—	0,00288	0,00052
3,4	—	—	—	—	—	0,00288	0,00080
3,6	—	—	—	—	—	0,00288	0,0012
4,0	—	—	—	—	—		

Continuation

$\dfrac{(8)}{1,27 \cdot 10^{-3}}$	$(9)^2$	$0,151 \cdot (10)$	$\dfrac{a_0 t}{l_1}$	$(12)-(11)$	$1-(13)$	$Q = 0,302 + (14)$
9	10	11	12	13	14	15
—	—	—	—	—	—	—
—	—	—	—	—	—	—
—	—	—	—	—	—	—
—	—	—	—	—	—	—
—	—	—	—	—	—	—
—	—	—	—	—	—	—
0,157	0,00387	0,000584	1,07	1,069	—0,069	0,233
0,315	0,0313	0,00473	1,11	1,135	—0,135	0,167
0,???	0,0584	0,0103	1,21	1,199	—0,199	0,103
0,630	0,250	0,0377	1,28	1,242	—0,242	0,060
0,946	0,841	0,128	1,43	1,362	—0,362	0,060

§11. THE DIRECTION BETWEEN A SHOCK WAVE AND A PARTITION OF FINITE THICKNESS

As already mentioned earlier, an estimate of the displacements and the strains arising in a structure in the case of an underwater explosion can be performed on the basis of analysis of the propagation of elastoplastic waves.

Another approach to the solution of this problem is also possible, wherein the structure is regarded as a mechanical system with a definite number of degrees of freedom and the motion is studied in generalized Lagrange coordinates. This approach presupposes that the velocity of propagation of the elastoplastic waves is infinitely large, so that these are excluded from consideration.

The motion of the system is determined by its mechanical parameters (mass, stiffness) and by the external load produced by the shock

wave. It is essential to clarify the difference in the final results
obtained by using the indicated two methods, and to establish what er-
ror arises in the case when the structure is regarded as a mechanical
system.

We shall investigate this problem using as a simple example nor-
mal incidence of a plane shock wave on an unbounded plate of finite
thickness.

As is well known, a reflected and a refracted wave are formed at
the instant when a wave encounters a boundary separating two media.

The amplitudes of these waves are determined by the relations

$$\frac{p_{\text{отр}}}{p} = \frac{\rho_2 a_2 - \rho_1 a_1}{\rho_2 a_2 + \rho_1 a_1};$$ (1.241)

$$\frac{p_{\text{пр}}}{p} = \frac{2\rho_2 a_2}{\rho_2 a_2 + \rho_1 a_1},$$ (1.240)

where ρ_2 and a_2 are the density and velocity of sound in the second
medium, ρ_1 and a_1 are the density and velocity of sound in the first
medium.

We introduce the notation: k_{11} for the coefficient of reflection
of the wave in the first medium from the second medium and k_{12} for the
coefficient of refraction from the first medium into the second medium.

We then obtain

$$\left.\begin{array}{l} p_{\text{отр}}^{1-2} = k_{11}p, \\ p_{\text{пр}}^{1-2} = k_{12}p. \end{array}\right\}$$ (4.110)

The refracted wave propagating in the second medium causes on
reaching the separation boundary between the second and third media
the formation of two additional waves, one reflected and the other re-
fracted (Fig. 102).

In this case, obviously,

$$p_{\text{отр}}^{2-3} = k_{12}p\,\frac{\rho_3 a_3 - \rho_2 a_2}{\rho_3 a_3 + \rho_2 a_2} = k_{12}k_{22}p,$$ (4.111)

where k_{22} is the coefficient of reflection of the wave in the second

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medium from the third medium;

$$p_{\text{пр}}^{2-3} = k_{12}p\,\frac{2\rho_3 a_3}{\rho_3 a_3 + \rho_2 a_2} = k_{12}k_{23}p, \qquad (4.112)$$

where k_{23} is the coefficient of refraction of the wave from the second medium into the third medium.
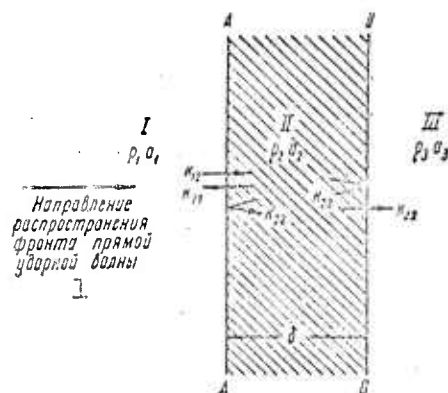

Fig. 102. Diagram showing the reflection and refraction of a shock wave in the case of normal incidence on a partition of finite thickness. 1) Direction of propagation of the front of the direct shock wave.

The wave reflected from the boundary separating the second and third media produces in turn, on reaching the boundary between the second and first media, a refracted wave which propagates in the first medium, as well as a wave reflected in the second medium.

Let us determine the parameters of these waves.

The pressure in the refracted wave, propagating in the first medium, will be

$$p_{\text{пр}}^{2-1} = p_{\text{отр}}^{2-3}\,\frac{2\rho_1 a_1}{\rho_1 a_1 + \rho_2 a_2} = k_{12}\bar{k}_{22}k_{21}p, \qquad (4.113)$$

where k_{21} is the coefficient of refraction from the second medium into the first.

The pressure in the reflected wave is

$$p_{\text{отр}}^{2-1} = p_{\text{отр}}^{2-3}\,\frac{\rho_1 a_1 - \rho_2 a_2}{\rho_1 a_1 + \rho_2 a_2} = k_{12}k_{22}\bar{k}_{22}p, \qquad (4.114)$$

where $\bar{k}_{22}$ is the coefficient of reflection of the wave in the second medium from the first medium.

The subsequent development of the wave process will be analogous to that already considered.

The pressure on the surfaces of the plate will change abruptly at time intervals equal to the duration during which the wave travels through double the thickness of the plate.

Using the foregoing considerations, let us calculate the pressure on the surface of the plate. The law governing the variation of the pressure in the direct wave will be assumed exponential:

$$p = p_m e^{-\frac{t}{\theta}} \sigma_0(t). \tag{4.115}$$

Denoting by $t_1 = \delta/a_2$ the travel time of the wave through the thickness of the partition, we obtain

$$
\begin{aligned}
p_1 = {}& \left(p_m e^{-\frac{t}{\theta}} + k_{11} p_m e^{-\frac{t}{\theta}}\right) \sigma_0(t) + p_m k_{12} k_{22} k_{21} \Big[e^{-\frac{t-2t_1}{\theta}} \sigma_0(t-2t_1) + \\
& + k_{22}\bar{k}_{22} e^{-\frac{t-4t_1}{\theta}} \sigma_0(t-4t_1) + (k_{22}\bar{k}_{22})^2 e^{-\frac{t-6t_1}{\theta}} \sigma_0(t-6t_1) + \\
& + (k_{22}\bar{k}_{22})^{n-1} e^{-\frac{t-2nt_1}{\theta}} \sigma_0(t-2nt_1) + \ldots \Big]
\end{aligned}
\tag{4.116}
$$

We introduce the additional notation:

$$
\left.
\begin{aligned}
\bar{a} &= k_{12}k_{22}k_{21}, \\
\gamma &= k_{22}\bar{k}_{22}, \\
\frac{t}{\theta} &= \tau, \\
\frac{t_1}{\theta} &= \tau_1
\end{aligned}
\right\}
\tag{4.117}
$$

We rewrite (4.116) in the form

$$p_1 = p_m(1+k_{11}) e^{-\tau} \sigma_0(\tau) + p_m \bar{a} \sum_{n=1}^{n=\infty} \gamma^{n-1} e^{-(\tau-2n\tau_1)} \sigma_0(\tau-2n\tau_1), \tag{4.118}$$

or, summing the series,

$$p_1 = p_m(1+k_{11}) e^{-\tau} \sigma_0(\tau) + p_m \bar{a} e^{-(\tau-2\tau_1)} \frac{\gamma^n e^{2n\tau_1}-1}{\gamma e^{2\tau_1}-1} \sigma_0(\tau-2\tau_1), \tag{4.119}$$

where $\underline{n}$ is an integer determined in accordance with the time interval under consideration from the inequality

$$2n\tau_1 \leqslant \tau \leqslant (2n+2)\tau_1.$$

A formula of this type was obtained by B. V. Zamyshlyayev and K. V. Lopukhov for the resultant load induced by an underwater shock wave on a partition of finite thickness.

The first term of Formula (4.119) represents the summary pressure of the direct and reflected waves. In the case when the acoustic impedance of the plate exceeds the acoustic impedance of the first and

third media, the remaining parts of Eq. (4.119) characterize the pressures of the rarefaction waves that emerge in succession from the second medium into the first.*

Arguing in similar fashion, we can readily obtain the pressure behind the plate:

$$p_3 = p_m k_{12} k_{23} \left\{ e^{-(\tau-\tau_1)} \sigma_0 (\tau - \tau_1) + \gamma e^{-(\tau-3\tau_1)} \sigma_0 (\tau - 3\tau_1) + \right.$$
$$+ \gamma^{n-1} e^{-[\tau-(2n-1)\tau_1]} \sigma_0 [\tau - (2n-1)\tau_1] + \ldots \} =$$
$$= p_m k_{12} k_{23} \sum_{n=1}^{n=\infty} \gamma^{n-1} e^{-[\tau-(2n-1)\tau_1]} \sigma_0 [\tau - (2n-1)\tau_1], \qquad (4.120)$$

or, putting $k_{12} k_{23} = \alpha$ and summing the series (4.120)

$$p_3 = p_m \alpha e^{-(\tau-\tau_1)} \frac{\gamma^n e^{2n\tau_1} - 1}{\gamma e^{2\tau_1} - 1} \sigma_0 (\tau - \tau_1), \qquad (4.121)$$

with

$$(2n-1) \tau_1 \leqslant \tau \leqslant (2n+1) \tau_1.$$

Of great practical interest is the determination of the parameters of the wave behind the partition, when both the first and the third medium is water. The solution of this problem is equivalent to estimating the shielding properties of solid partitions of specified thickness.

In this case

$$\rho_1 a_1 = \rho_3 a_3 = \rho_0 a_0;$$
$$\alpha = k_{12} k_{23} = \frac{2\rho_2 a_2}{\rho_2 a_2 + \rho_0 a_0} \frac{2\rho_0 a_0}{\rho_0 a_0 + \rho_2 a_2} = \frac{4\rho_0 a_0 \rho_2 a_2}{(\rho_0 a_0 + \rho_2 a_2)^2}; \qquad (4.122)$$
$$\gamma = \left(\frac{\rho_0 a_0 - \rho_2 a_2}{\rho_0 a_0 + \rho_2 a_2} \right)^2. \qquad (4.123)$$

From a comparison of (4.122) and (4.123) we arrive at the conclusion that $\alpha + \gamma = 1$.

In accordance with (4.121) and (4.178), the value of the impulse of the n-th portion of the pressure pattern is equal to

$$I_n = p_m 0 \alpha \frac{\gamma^n e^{2n\tau_1} - 1}{\gamma e^{2\tau_1} - 1} \int_{(2n-1)\tau}^{(2n+1)\tau_1} e^{-(\tau-\tau_1)} a\tau, \qquad (4.124)$$

or, inasmuch as

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$$\int_{(2n-1)\tau_1}^{(2n+1)\tau_1} e^{-(\tau-\tau_1)}\,d\tau = e^{-2n\tau_1}(e^{2\tau_1}-1),$$

we have .

$$I_n = p_m 0\alpha\,\frac{\gamma^n e^{2n\tau_1}-1}{\gamma e^{2\tau_1}-1}\,e^{-2n\tau_1}(e^{2\tau_1}-1) =$$

$$= \frac{p_m 0\alpha}{\gamma e^{2\tau_1}-1}(e^{2\tau_1}-1)\left\{(\gamma^n e^{2n\tau_1}-1)\,e^{-2n\tau_1}\right\}. \qquad (4.125)$$

The total impulse can be obtained by summing the expressions (4.125) from $n = 1$ to $n = \infty$:

$$I = \sum_{n=1}^{n=\infty} I_n = \frac{p_m 0\alpha}{\gamma e^{2\tau_1}-1}(e^{2\tau_1}-1)\sum_{n=1}^{n=\infty}(\gamma^n - e^{-2n\tau_1}) =$$

$$= \frac{p_m 0\alpha}{\gamma e^{2\tau_1}-1}(e^{2\tau_1}-1)\left\{\frac{\gamma}{1-\gamma}-\frac{e^{-2\tau_1}}{1-e^{-2\tau_1}}\right\} =$$

$$= \frac{p_m 0\alpha}{\gamma e^{2\tau_1}-1}(e^{2\tau_1}-1)\,\frac{\gamma - e^{-2\tau_1}}{(1-\gamma)(1-e^{-2\tau_1})} \qquad (4.126)$$

or, canceling out and recognizing that $\alpha = 1 - \gamma$, we obtain ultimately

$$I = p_m 0. \qquad (4.127)$$

Thus, independently of the acoustic properties and the thickness of the elastic partition, the impulse of the pressures behind the partition is equal to the total impulse of the direct shock wave. The role of the partition reduces, as it were, to an increase in the time of action of the positive phase of the pressure while decreasing the value of the maximum pressure, which depends linearly on the coefficient α. Therefore, in order to reduce appreciably the maximum pressure it is necessary to construct partitions with acoustic impedances that are much smaller than the acoustic impedance of the surrounding medium.

Let us return to the estimate of the resultant pressure on the front surface of the plate.

We have shown earlier that this pressure is determined by the expression

$$p_1 = p_m(1 + k_{11})e^{-\tau}\sigma_0(\tau) + p_m\bar{a}e^{-(\tau - 2\tau_1)}\frac{\gamma^n e^{2n\tau_1} - 1}{\gamma e^{\tau_1} - 1}\sigma_0(\tau - 2\tau_1). \qquad (4.119)$$

We obtain an analogous relation by considering the plate as a mechanical system with one degree of freedom. Assuming the plate to be absolutely rigid ($a_2 = \infty$, $k_{11} = 1$), we shall take into account only its inertial properties.

As soon as the front of the plane shock wave touches the surface of the plate, the latter begins to move like a solid, with velocity dz/dt. This displacement causes the formation of a pressure field behind the partition, which in the acoustic approximation is characterized by the quantity

$$p_3 = \rho_3 a_3 \frac{dz}{dt}. \qquad (4.128)$$

The equation of motion of the plate is written in the form

$$m\frac{d^2z}{dt^2} = p(t) + p_{\text{отр}}(t) - p_3(t), \qquad (4.129)$$

where $\underline{m}$ is the mass of the plate per unit surface, p(t) the pressure in the direct wave, $p_{\text{отр}}(t)$ the pressure in the reflected wave, and $p_3(t)$ the pressure in the wave behind the plate.

The pressure in the direct and reflected waves are related by the condition that the velocities of the particles of the first medium must on the surface of the partition be equal to the velocity of the partition itself. As a result of this

$$p_{\text{отр}}(t) = p(t) - \rho_1 a_1 \frac{dz}{dt}. \qquad (4.130)$$

Thus

$$m\frac{d^2z}{dt^2} + (\rho_1 a_1 + \rho_3 a_3)\frac{dz}{dt} = 2p(t). \qquad (4.131)$$

For a wave with exponential profile we have

$$m\frac{d^2z}{dt^2} + (\rho_1 a_1 + \rho_3 a_3)\frac{dz}{dt} = 2p_m e^{-\frac{t}{\theta}}. \qquad (4.132)$$

A general solution of (4.132) is

$$z = c_1 + c_2 e^{-\beta \frac{t}{\theta}} - \frac{2 p_m \theta^2}{m(\beta - 1)} e^{-\frac{t}{\theta}}, \tag{4.133}$$

where

$$\beta = \frac{\rho_1 a_1 + \rho_3 a_3}{m} \theta. \tag{4.134}$$

The integration constants c_1 and c_2 are determined from the conditions:

when $t = 0$ and $z = 0$ we have

$$dz/dt = 0.$$

After elementary transformations, we obtain

$$z = \frac{2 p_m \theta^2}{m\beta(\beta - 1)} \left[\beta - 1 + e^{-\frac{\beta t}{\theta}} - \beta e^{-\frac{t}{\theta}} \right], \tag{4.135}$$

hence

$$\frac{dz}{dt} = \frac{2 p_m \theta}{m(\beta - 1)} \left[e^{-\frac{t}{\theta}} - e^{-\frac{\beta t}{\theta}} \right]. \tag{4.136}$$

The resultant pressure on the front surface of the plate is equal to

$$p_1 = p(t) + p_{\text{отр}}(t) = 2p(t) - \rho_1 a_1 \frac{dz}{dt} =$$

$$= 2 p_m e^{-\frac{t}{\theta}} - \rho_1 a_1 \frac{2 p_m \theta}{m(\beta - 1)} \left[e^{-\frac{t}{\theta}} - e^{-\frac{\beta t}{\theta}} \right] =$$

$$= \frac{2 p_m}{\beta - 1} \left\{ (\beta - 1) e^{-\frac{t}{\theta}} - \rho_1 a_1 \frac{\theta}{m} e^{-\frac{t}{\theta}} + \rho_1 a_1 \frac{\theta}{m} e^{-\frac{\beta t}{\theta}} \right\}, \tag{4.137}$$

or, putting,

$$\rho_1 a_1 \frac{\theta}{m} = \beta_1; \tag{4.138}$$

$$\beta_1 + \beta_3 = \beta; \tag{4.139}$$

$$\rho_3 a_3 \frac{\theta}{m} = \beta_3, \tag{4.140}$$

we obtain

$$p_1 = \frac{2 p_m}{\beta - 1} \left[\beta_1 e^{-\frac{\beta t}{\theta}} - (1 - \beta_3) e^{-\frac{t}{\theta}} \right]. \tag{4.141}$$

The pressure behind the plate is

$$p_3 = \rho_3 a_3 \frac{dz}{dt} = \rho_3 a_3 \frac{\theta}{m} \frac{2 p_m}{\beta - 1} \left[e^{-\frac{t}{\theta}} - e^{-\frac{\beta t}{\theta}} \right] =$$

$$= \frac{\beta_3}{\beta - 1} 2 p_m \left[e^{-\frac{t}{\theta}} - e^{-\frac{\beta t}{\theta}} \right]. \tag{4.142}$$

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In the case when the acoustic impedance of the third medium is much smaller than that of the first ($\rho_3 a_3 / \rho_1 a_1 \ll 1$), and when, consequently, $\beta_1 \simeq \beta$, we have in lieu of (4.141)

$$p_i = \frac{2p_m}{\beta - 1}\left[\beta e^{-\frac{\beta t}{\theta}} - e^{-\frac{t}{\theta}}\right]. \tag{4.143}$$

It can be shown that the approximate solution (4.137), which does not take into account the wave character of the propagation of the disturbances, gives for the overwhelming majority of practical problems a result which is close to the accurate one, and has a much simpler form.

For the reason indicated, one customarily uses Relations (4.137)-(4.142), in spite of the fact that Formulas (4.119) and (4.120) take a fuller account of the wave processes that occur in the finite-thickness partition itself.

Example 1. Calculate the pressure on the front surface of a steel plate for the incidence of a plane shock wave of exponential form. The plate separates air from water. The shock wave is produced underground.

Solve the problem with the aid of the approximate and exact relations.

For the initial data assume the following:

$$\rho_1 = 102 \ \text{kg-sec}^2/\text{m}^4 \qquad\qquad a_1 = 1500 \ \text{m/sec},$$
$$\rho_2 = 800 \ \text{kg-sec}^2/\text{m}^4, \qquad\qquad a_2 = 5500 \ \text{m/sec},$$
$$\rho_3 = 0.125 \ \text{kg-sec}^2/\text{m}^4, \qquad\qquad a_3 = 340 \ \text{m/sec}.$$

The exponential-attenuation constant is $\theta = 1.82 \cdot 10^{-5}$ sec. The pressure on the front is $\rho_m = 100 \ \text{kg/cm}^2$. The thickness of the plate is $\delta = 0.5$ cm.

Solution. We calculate the values of the coefficients contained in Formula (4.118):

$$k_{11} = \frac{\rho_2 a_2 - \rho_1 a_1}{\rho_2 a_2 + \rho_1 a_1} = \frac{800 \cdot 5500 - 102 \cdot 1500}{800 \cdot 5500 + 102 \cdot 1500} = 0.933;$$

$$k_{22} = \frac{\rho_3 a_3 - \rho_2 a_2}{\rho_3 a_3 + \rho_2 a_2} = \frac{0.125 \cdot 340 - 800 \cdot 5500}{0.125 \cdot 340 + 800 \cdot 5500} \approx -1.00;$$

$$\bar{k}_{22} = \frac{\rho_1 a_1 - \rho_2 a_2}{\rho_1 a_1 + \rho_2 a_2} = \frac{102 \cdot 1500 - 800 \cdot 5500}{102 \cdot 1500 + 800 \cdot 5500} = -0,933;$$

$$\gamma = k_{22} \cdot \bar{k}_{22} = 0,933;$$

$$k_{12} = \frac{2\rho_2 a_2}{\rho_2 a_2 + \rho_1 a_1} = \frac{2 \cdot 800 \cdot 5500}{800 \cdot 5500 + 102 \cdot 5500} = 1,935;$$

$$k_{21} = \frac{2\rho_1 a_1}{\rho_1 a_1 + \rho_2 a_2} = \frac{2 \cdot 102 \cdot 1500}{102 \cdot 1500 + 800 \cdot 5500} = 0,0673;$$

$$\bar{a} = k_{12} \cdot k_{22} \cdot k_{21} = 1,935 - 1,00 \cdot 0,0673 = -0,130;$$

$$\tau_1 = \frac{t_1}{0} = \frac{\delta}{a_2 0} = \frac{0,5}{5500 \cdot 10^2 \cdot 1,82 \cdot 10^{-6}} = 0,05.$$

After substituting these values in (4.118) we obtain

$$p_1 = p_m \cdot 1,933 e^{-\tau} \sigma_0(\tau) + p_m(-0,130) e^{-(\tau - 2\tau_1)} \sigma_0(\tau - 2\tau_1) +$$
$$+ p_m(-0,130) 0,933 e^{-(\tau - 4\tau_1)} \sigma_0(\tau - 4\tau_1) +$$
$$+ p_m(-0,130) 0,871 e^{-(\tau - 6\tau_1)} \sigma_0(\tau - 6\tau_1) + \ldots$$

Calculations by means of this formula are conveniently carried out in tabular form, choosing as the time interval the quantity $\tau = 2\tau_1$, as is done in Table 28. After such a time interval, rarefaction waves will arrive at the front surface of the plate, and the pressures will vary abruptly (Fig. 103).
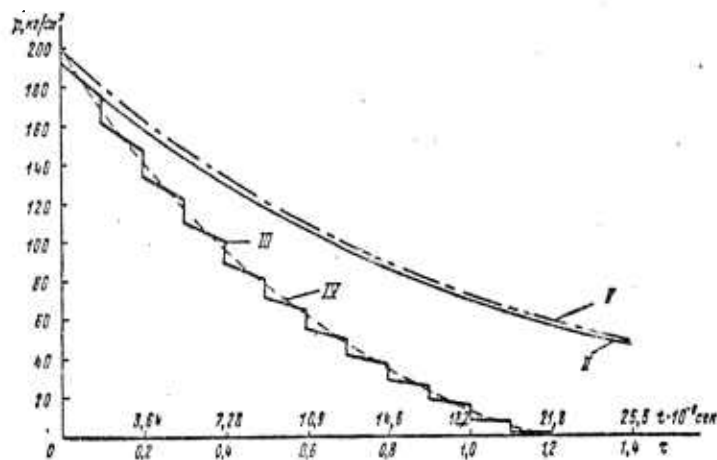
Fig. 103. Pressure on the surface of a plate. I) Pressure on an absolutely rigid plate; II) pressure on a plate of infinite thickness; III) pressure on a plate of finite thickness in accordance with the exact equation; IV) pressure on a plate of finite thickness in accordance with the approximate equation.

We now solve the same problem using the approximate method, regarding the plate as a mechanical system with one degree of freedom.

Let us calculate the parameters entering into Formula (4.143):

$$\beta = \frac{\frac{102 \cdot 1500 - 0.125 \cdot 340}{8 \cdot 5}}{9.81 \cdot 10^4} 1.52 \cdot 10^{-3} = 0.685;$$

$$p_1 = \frac{2 \cdot 100}{1 - 0.685}(e^{-\tau} - 0.685 e^{-0.685\tau}) = 635\,(e^{-\tau} - 0.685 e^{-0.685\tau}). \qquad (4.143)$$

The results of the calculations are given in Table 28.

TABLE 28

n	$\tau=2n\tau_1$	$e^{-\tau}$	$e^{-\tau}\cdot 1.93$	γ^{n-1}	$(5)\cdot\bar{a}$	$\beta\tau$	$\beta e^{-\beta\tau}$	$(3)-(8)$	$P_1=635\cdot(9)$	$\beta\tau$	$e^{-\beta\tau}$	$e^{-\tau}-e^{-\beta\tau}$	$P=370\cdot(13)$
1	2	3	4	5	6	7	8	9	10	11	12	13	14
0	0	1.00	1.93	—	—	0	0.685	0.315	2.00	0	1	0	0
1	0.10	0.904	1.74	1.000	−0.130	0.0685	0.640	0.264	1.68				
2	0.20	0.820	1.58	0.933	−0.126	0.137	0.596	0.224	1.43	0.274	0.760	0.060	22.2
3	0.30	0.740	1.43	0.871	−0.113	0.205	0.557	0.183	1.17				
4	0.40	0.671	1.27	0.812	−0.1055	0.274	0.521	0.150	0.956	0.548	0.578	0.093	34.4
5	0.50	0.606	1.17	0.758	−0.0985	0.342	0.486	0.120	0.765				
6	0.60	0.550	1.06	0.708	−0.0920	0.412	0.454	0.096	0.612	0.822	0.439	0.111	41.2
7	0.70	0.496	0.957	0.662	−0.0860	0.478	0.425	0.071	0.454				
8	0.80	0.450	0.868	0.617	−0.0803	0.548	0.396	0.054	0.341	1.095	0.334	0.116	42.8
9	0.90	0.406	0.784	0.575	−0.0746	0.616	0.370	0.036	0.229				
10	1.00	0.368	0.710	0.537	−0.0698	0.685	0.346	0.022	0.140	1.37	0.254	0.114	42.2
11	1.10	0.334	0.644	0.501	−0.0650	0.754	0.322	0.012	0.0765				
12	1.20	0.296	0.572	0.463	−0.0600	0.823	0.301	−0.005	−0.0320	1.64	0.194	0.102	36.8
13	1.30	0.273	0.525	0.426	−0.0566	0.891	0.281	−0.008	−0.0510				
14	1.40	0.248	0.478	0.407	−0.0550	0.958	0.262	−0.014	−0.0821	1.92	0.146	0.101	37.3
15	1.50	0.223	0.400	0.000	−0.0491	1.028	0.246	−0.022	−0.140				

A comparison of the data obtained by these two methods is shown in Fig. 103.

Example 2. Determine the pressure behind a steel plate of thickness $\delta = 0.5$ cm, immersed in water, acted upon by an underwater shock wave with maximum pressure $p_m = 100$ kg/cm^2 and an attenuation time constant $\theta = 1.82 \cdot 10^{-5}$ sec.

Compare the results of the calculations obtained with the exact and with the approximate formulas.

Solution. We calculate the coefficients γ and $k_{12}k_{23}$ contained in Formula (4.128):

$$k_{12}k_{23} = \frac{2\rho_2 a_2}{\rho_2 a_2 + \rho_1 a_1} \frac{2\rho_1 a_1}{\rho_1 a_1 + \rho_2 a_2} = \frac{4\rho_1 a_1 \rho_2 a_2}{(\rho_1 a_1 + \rho_2 a_2)^2} =$$

$$= \frac{4 \cdot 102 \cdot 1500 \cdot 800 \cdot 5500}{(102 \cdot 1500 + 800 \cdot 5500)^2} \approx 0{,}130;$$

$$\gamma = k_{22}\bar{k}_{22} = \frac{\rho_3 a_3 - \rho_2 a_2}{\rho_3 a_3 + \rho_2 a_2} \frac{\rho_1 a_1 - \rho_2 a_2}{\rho_1 a_1 + \rho_2 a_2} = \frac{(\rho_1 a_1 - \rho_2 a_2)^2}{(\rho_1 a_1 + \rho_2 a_2)^2} =$$

$$= \frac{(102 \cdot 1500 - 800 \cdot 5500)^2}{(102 \cdot 1500 + 800 \cdot 5500)^2} \approx 0{,}870.$$

Since the time interval $\tau = 2n\tau_1$ remains the same as before, we can use the results of the calculations given in Table 28 for the calculation of the pressure pattern. The pressure behind the plate occurs after the shock wave covers a distance equal to the thickness of the plate, and will then change abruptly every $2n\tau_1$.
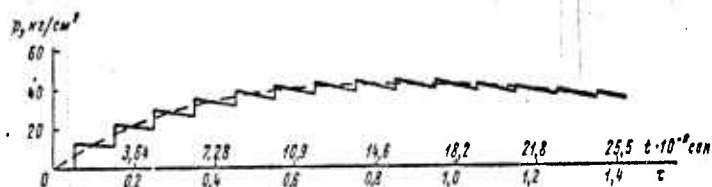

Fig. 104. Pressure behind a plate: solid line — using the exact formula; dashed — using the approximate formula.

An approximate estimate of the character of pressure variation can be made using Eq. (4.142), and in our case

$$\beta_1 = \frac{\rho_1 a_1}{m} 0 = \frac{102 \cdot 1500}{\frac{8 \cdot 0{,}5}{9{,}81 \cdot 10^5}} 1{,}82 \cdot 10^{-5} = 0{,}685;$$

$$\beta = \beta_1 + \beta_3 = 2\beta_1 = 1{,}37,$$

therefore

$$p_3 = \frac{0{,}685}{1{,}37 - 1{,}00} 2P_m \left[e^{-\tau} - e^{-1{,}37\tau} \right] = 370 \left[e^{-\tau} - e^{-1{,}37\tau} \right].$$

Calculation by means of this formula is made in Table 28.

The results of the comparison of the data obtained by the two methods are shown in Fig. 104.

§12. DYNAMIC DESIGN OF A SYSTEM WITH ONE DEGREE OF FREEDOM FOR THE ACTION OF AN UNDERWATER SHOCK WAVE

The solution of the problem of dynamic design of a system to withstand the action of an underwater shock wave is, as already mentioned,

very difficult in the general formulation. We therefore consider here the following two limiting cases:

1) the dimensions of the structure are so large that the time of arrival of the diffraction wave from the edges of the partition exceeds the duration of the compression phase of the direct shock wave;

2) the dimensions of the structure are so small that the time of arrival of the diffraction wave from the edges of the partition is appreciably shorter than the duration of the compression phase of the direct wave.

In the former case we can neglect the diffraction phenomena. Using the reduction method, we obtain the differential equation of motion of the system in the form

$$\rho_c \delta \frac{d^2 z}{dt^2} + (\rho_1 a_1 + \rho_3 a_3) \frac{dz}{dt} + kz = 2p_m e^{-\frac{t}{\theta}}. \qquad (4.144)$$

Equation (4.144) differs from (4.132) only in the term that takes into account the stiffness of the structure.

Of basic principal interest is the case when air is located behind the cover or behind a plate. In this case $\rho_3 a_3 \ll \rho_1 a_1$ and without loss of practical accuracy we have in lieu of (4.144)

$$\rho_c \delta \frac{d^2 z}{dt^2} + \rho_1 a_1 \frac{dz}{dt} + kz = 2p_m e^{-\frac{t}{\theta}}. \qquad (4.145)$$

The problem in this formulation was considered in detail by V.V. Novozhilov, D.A. Aleksandrin, and M.N. Lefonova.

We put

$$\omega^2 = \frac{k}{\rho_c \delta}; \qquad (4.146)$$

$$\beta = \frac{\rho_1 a_1}{\rho_c \delta} \theta, \qquad (4.147)$$

and then we obtain

$$\ddot{z} + \frac{\beta}{\theta} \dot{z} + \omega^2 z = \frac{2p_m}{\rho_c \delta} e^{-\frac{t}{\theta}}. \qquad (4.148)$$

The general solution of Eq. (4.148) is

$$z = c_1 e^{-\alpha_1 \frac{t}{\theta}} + c_2 e^{-\alpha_2 \frac{t}{\theta}} + \frac{2p_m \theta^2}{(1 - \beta + \omega^2 \theta^2)} \frac{1}{\rho_c \delta} e^{-\frac{t}{\theta}}, \qquad (4.149)$$

where

$$\alpha_1 = \frac{\beta}{2}(1 + \sqrt{1 - \xi^2}); \qquad (4.150)$$

$$\alpha_2 = \frac{\beta}{2}(1 - \sqrt{1 - \xi^2}); \qquad (4.151)$$

$$\xi^2 = \frac{4\theta^2 \omega^2}{\beta^2}. \qquad (4.152)$$

Subjecting Eq. (4.149) to the initial conditions $z = dz/dt = 0$ when $t = 0$, we obtain

$$z = \frac{2p_m \theta^2}{\rho_c \delta (1 - \beta + \omega^2 \theta^2)} \frac{1}{\alpha_1 - \alpha_2} \Big[-(1 - \alpha_2) e^{-\alpha_1 \frac{t}{\theta}} +$$
$$+ (1 - \alpha_1) e^{-\alpha_2 \frac{t}{\theta}} + (\alpha_1 - \alpha_2) e^{-\frac{t}{\theta}} \Big]. \qquad (4.153)$$

In most practical cases $\xi^2 \ll 1$. We can therefore assume

$$\alpha_1 \simeq \beta\left(1 - \frac{1}{4}\xi^2\right) \simeq \beta; \qquad (4.154)$$

$$\alpha_2 \simeq \frac{1}{4}\beta\xi^2\left(1 + \frac{1}{4}\xi^2\right) \simeq \frac{1}{4}\beta\xi^2; \qquad (4.155)$$

$$\alpha_1 - \alpha_2 = \beta\left(1 - \frac{1}{2}\xi^2\right).$$

Substituting these values of α_1 and α_2 in (4.153), and taking into consideration the fact that

$$\theta^2 \omega^2 = \frac{\xi^2 \beta^2}{4},$$

we obtain

$$z(t) = \frac{2p_m \theta^2}{\rho_c \delta} \frac{1}{1 - \beta + \frac{\beta^2 \xi^2}{4}} \frac{1}{\beta\left(1 - \frac{\xi^2}{2}\right)} \Big\{ -\left(1 - \frac{\beta}{4}\xi^2\right) e^{-\beta \frac{t}{\theta}} +$$
$$+ (1 - \beta) e^{-\frac{1}{4}\beta\xi^2 \frac{t}{\theta}} + \beta\left(1 - \frac{1}{2}\xi^2\right) e^{-\frac{t}{\theta}} \Big\}. \qquad (4.156)$$

Differentiating this expression, we have

$$\dot{z} = \frac{2p_m \theta}{\rho_c \delta \left(1 - \beta + \frac{\beta^2 \xi^2}{4}\right)} \frac{1}{1 - \frac{\xi^2}{2}} \Big\{ \left(1 - \frac{\beta}{4}\xi^2\right) e^{-\beta \frac{t}{\theta}} -$$
$$- \frac{1}{4}\xi^2(1 - \beta) e^{-\frac{\beta}{4}\xi^2 \frac{t}{\theta}} - \left(1 - \frac{1}{2}\xi^2\right) e^{-\frac{t}{\theta}} \Big\}. \qquad (4.157)$$

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Equating $z(t) = 0$ and determining from the resultant transcendental equation the time t, we can obtain the following approximate expression for the maximum value of the displacement, valid for $\beta \geq 5$:

$$z_{max} \simeq \frac{2p_m 0}{\rho_1 a_1} \left\{ \frac{\frac{1}{4}\xi^2\left[\beta\left(1 - \frac{1}{4}\xi^2\right) - 1\right]}{1 - \frac{1}{2}\xi^2} \right\}^{\frac{\frac{1}{4}\beta\xi^2}{1 - \frac{1}{4}\beta\xi^2}}. \qquad (4.158)$$

Let us now consider a second case, when the dimensions of the structure are small compared with the length of the shock wave. In this case the diffraction waves from the edges of the partition will rapidly eliminate the pressure in the reflected wave. The displacements of the structure itself lead to a change of the hydrodynamic fields, which can be taken into account with good approximating by introducing the apparent mass coefficient. Therefore the differential equation of motion of the system will in this case have the form

$$m\ddot{z} + kz = p(t), \qquad (4.159)$$

where m is the sum of the mass of the plate and its apparent mass, per unit surface.

Since equations of the type (4.159) with an exponential function in the right half have been investigated in great detail, further calculations will be carried out under the assumption that the form of the shock wave is specified by the hyperbolic relation

$$p(t, r) = p_m \frac{1}{\left[\alpha\left(\frac{ta_0}{R_{03}} - \frac{r}{R_{03}}\right) + 1\right]^2} \sigma_0\left(\frac{ta_0}{R_{03}} - \frac{r}{R_{03}}\right). \qquad (4.160)$$

As was already mentioned earlier, such a specification of the function $p(t, r)$ is no less accurate than the exponential form, and in individual cases it is more convenient.

Taking as the initial instant of time the time of arrival of the shock wave at the structure, we obtain

$$\ddot{z} + \omega^2 z = \frac{p_m}{m_e} \frac{1}{\left[a \frac{t a_0}{R_{03}} - 1 \right]^2},$$ (4.161)

where

$$\omega^2 = \frac{k}{m_e}.$$

A solution of this equation (4.161) will be

$$z = \frac{p_m}{m} \frac{1}{\omega} \int_0^t \frac{\sin \omega (t - \tau)}{\left[a \frac{a_0}{R_{03}} \tau + 1 \right]^2} d\tau.$$ (4.162)

We introduce a change of variable

$$\tau = \frac{x}{\omega} - \frac{R_{03}}{a a_0}.$$ (4.163)

We have

$$z = \frac{p_m}{m} \frac{R_{03}}{a^2 a_0^2} \int_{\frac{\omega R_{03}}{a a_e}}^{\omega t + \frac{\omega R_{03}}{a a_e}} \frac{\sin \left(\omega t - x + \frac{\omega R_{03}}{a a_0} \right)}{x^2} dx.$$ (4.164)

For convenience we write for the dimensionless combination

$$\omega t + \frac{\omega R_{03}}{a a_0} = \xi.$$ (4.165)

We then obtain in lieu of (4.164)

$$\dot{z} = \frac{p_m}{m} \frac{R_{03}^2}{a^2 a_0^2} \int_{\frac{\omega R_{03}}{a a_e}}^{\xi} \frac{\sin (\xi - x)}{x^2} dx.$$ (4.166)

Differentiating with respect to $\underline{t}$, we obtain

$$\dot{z} = \frac{p_m}{m} \frac{R_{03}^2}{a^2 a_0^2} \omega \int_{\frac{\omega R_{03}}{a a_e}}^{\xi} \frac{\cos (\xi - x)}{x^2} dx.$$ (4.167)

The integrals in the right halves of (4.166) and (4.167) are determined, as is well known, with the aid of special functions, the integral sine and cosine.

After simple transformations we have

$$z = \frac{p_m}{m} \frac{R_{03}^2}{a^2 a_0^2} \left\{ \sin \xi \left[\Phi_1 \left(\frac{\omega R_{03}}{a a_\Lambda} \right) - \Phi_1 (\xi) \right] \right\}.$$

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$$-\cos\xi\left[\Phi_2\left(\frac{\omega R_{03}}{aa_0}\right)-\Phi_2\left(\xi\right)\right]\right\},\tag{4.168}$$

$$\dot{z}=\frac{p_m}{m}\frac{R_{03}^2}{a^2a_0^2}\,\omega\left\{\cos\xi\left[\Phi_1\left(\frac{\omega R_{03}}{aa_0}\right)-\Phi_1\left(\xi\right)\right]+\right.$$
$$\left.+\sin\xi\left[\Phi_2\left(\frac{\omega R_{03}}{aa_0}\right)-\Phi_2\left(\xi\right)\right]\right\},\tag{4.169}$$

where

$$\xi=\omega\left(t+\frac{R_{03}}{aa_0}\right);\tag{4.170}$$

$$\Phi_1\left(\xi\right)=\frac{\cos\xi}{\xi}+\mathrm{si}\,\xi;\tag{4.171}$$

$$\Phi_2\left(\xi\right)=\frac{\sin\xi}{\xi}-\mathrm{ci}\,\xi.\tag{4.172}$$

To determine the argument ξ, which corresponds to the instant of the extremum of z in Eq. (4.168), it is sufficient to set $\dot{z}$ equal to zero. We then arrive at the transcendental equation

$$\cos\xi\Phi_1\left(\zeta\right)+\sin\xi\Phi_2\left(\zeta\right)=\cos\xi\Phi_1\left(\xi\right)+\sin\xi\Phi_2\left(\xi\right),\tag{4.173}$$

where ζ denotes

$$\zeta=\frac{\omega R_{03}}{aa_0}.\tag{4.174}$$

An arbitrarily assigned parameter ζ corresponds to an infinite set of values of ξ, which turn Eq. (4.173) into an identity. In the physical sense of the problem, we are interested in the value of ξ that is closest to ζ and satisfies Eq. (4.173). In addition, we should obviously have $\xi>\zeta$. Equation (4.173) is solved by a grapho-analytic method. If we substitute the function $\xi(\zeta)$ obtained in this manner into Eq. (4.168), we obtain the extremal value of z:

$$z_m=\frac{p_m}{m}\frac{R_{03}^2}{a^2a_0^2}\left\{\sin\xi\left(\zeta\right)\left[\Phi_1\left(\zeta\right)-\Phi_1\left[\xi\left(\zeta\right)\right]\right]-\right.$$
$$\left.-\cos\xi\left(\zeta\right)\left[\Phi_2\left(\zeta\right)-\Phi_2\left[\xi\left(\zeta\right)\right]\right]\right\},$$

or, since

$$R_{03}=\frac{\zeta aa_0}{\omega},\tag{4.174}$$

we have

$$z_m=\frac{p_m}{m}\frac{1}{\omega^2}\,F\left(\zeta\right),\tag{4.175}$$

where

$$F(\zeta) = \zeta^2 \{ \sin \xi(\zeta) [\Phi_1(\zeta) - \Phi_1[\xi(\zeta)]] - \cos \xi(\zeta) [\Phi_2(\zeta) - \Phi_2[\xi(\zeta)]] \}. \qquad (4.176)$$

The calculation and succeeding analytic approximation of the function $F(\zeta)$ lead, with good approximation, to the relation

$$F(\zeta) \simeq \frac{2\zeta}{\zeta + 2.4}. \qquad (4.177)$$

We therefore can write

$$z_m = \frac{p_m}{m} \frac{1}{\omega^2} \frac{2\zeta}{\zeta + 2.4}. \qquad (4.178)$$

If we recall that $\omega^2 = k/m$ and $p_m/k = z_{st}$ is the deflection that this plate would receive were p_m to be applied statically, then we obtain easily after elementary transformations

$$z_m = \frac{2\zeta}{\zeta + 2.4} z_{ct}. \qquad (4.179)$$

Thus, the coefficient $2\zeta/(\zeta + 2.4)$ is the dynamicity coefficient.

At large charge radii or at high frequencies ω we have

$$\frac{2\zeta}{\zeta + 2.4} \simeq 2,$$

i.e., the effect of a suddenly applied constant load is obtained.

To the contrary, at small ω and R_{03}, the dynamicity coefficient can be as small as desired.

Example 1. Calculate the maximum deflection of a steel plate in the outer hull of a ship with the following initial data:

$$\rho_1 = 102 \text{ kg-sec}^2/\text{m}^4, \qquad a_1 = 1500 \text{ m/sec};$$
$$\rho_2 = 800 \text{ kg-sec}^2/\text{m}^4, \qquad a_2 = 5500 \text{ m/sec}.$$

The pressure on the front of the underwater shock wave is $p_m = 120 \text{ kg/cm}^2$.

The thickness of the plate is $\delta = 0.8$ cm.

The stiffness coefficient is $k = 140 \text{ kg/cm}^3$, and the exponential damping constant is $\theta = 2.5 \cdot 10^{-4}$ sec.

<u>Solution.</u> We find the frequency of oscillation of the plate:

$$\omega^2 = \frac{k}{\rho_2 \delta} = \frac{140 \cdot 10^3}{800 \cdot 0.8} = 21.9 \cdot 10^3 \ 1/\text{sec}^2,$$

$$\omega = 4.66 \cdot 10^3 \ 1/\text{sec}.$$

We calculate the coefficients β and ξ

$$\beta = \frac{\rho_1 a_1}{\rho_2 \delta} \theta = \frac{102 \cdot 1500}{800 \cdot 0.8 \cdot 10^2} 2.51 \cdot 10^{-4} = 6.0;$$

$$\xi^2 = \frac{4\theta^2 \omega^2}{\beta^2} = \frac{4 \cdot 2.5^2 \cdot 10^{-8} \cdot 4.66^2 \cdot 10^6}{36} = 0.15.$$

The maximum deflection of the plate is determined from the formula

$$z_{\max} \simeq \frac{2 p_m \theta}{\rho_1 a_1} \left\{ \frac{\frac{1}{4} \xi^2 \left[\beta \left(1 - \frac{1}{4} \xi^2\right) - 1 \right]}{1 - \frac{1}{2} \xi^2} \right\}^{\frac{\frac{1}{4}\beta\xi^2}{1 - \frac{1}{4}\beta\xi^2}} =$$

$$= \frac{2 \cdot 120 \cdot 2.5 \cdot 10^{-4}}{102 \cdot 1500} 10^6 \left\{ \frac{\frac{1}{4} 0.15 \left[6.0 \left(1 - \frac{1}{4} 0.15\right) - 1 \right]}{1 - \frac{1}{2} 0.15} \right\}^{\frac{\frac{1}{4} 6.0 \cdot 0.15}{1 - \frac{1}{4} 6.0 \cdot 0.15}} =$$

$$= 0.392 \left\{ \frac{0.0375 \cdot 4.77}{0.925} \right\}^{0.29} = 0.392 \cdot 0.194^{0.29} = 0.244 \ \text{cm}.$$

<u>Example 2.</u> A copper membrane 50 mm in diameter and 3 mm thick is located at a distance of 100 m from the center of the charge, and was subjected to a deflection of 6.5 mm.

Find the weight of the charge if the following are specified: the stiffness coefficient of the membrane $k \simeq 250 \ \text{kg/cm}^3$; mass of the membrane per unit surface, with allowance for the apparent mass of the liquid, $m_* = 4.2 \cdot 10^{-6} \ \text{kg-sec}^2/\text{cm}^3$.

<u>Solution.</u> We obtain the frequency of the natural oscillations of the membrane:

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{250}{4.2 \cdot 10^{-6}}} = 7.7 \cdot 10^3 \ 1/\text{sec}.$$

According to (4.175) we have

$$z_m = \frac{p_m}{k} \frac{2\zeta}{\zeta + 2.4} = \frac{p_m}{k} \frac{2 \frac{\omega R_{03}}{a_0^2}}{\frac{\omega R_{03}}{a_0^2} + 2.4},$$

with

$$p_m = \frac{14\ 700}{\left(\frac{r}{R_{03}}\right)^{1.13}}.$$

Thus

$$z_m = \frac{14\,700}{\left(\frac{r}{R_{03}}\right)^{1,13}} \frac{1}{k} \frac{2\frac{\omega R_{03}}{a_0^2}}{\frac{\omega R_{03}}{a_0^2} + 2,4}.$$

Since the distance $\underline{r}$, the stiffness coefficient $\underline{k}$, the deflection of the membrane Z_m, and the oscillation frequency ω are specified, the last equation enables us to determine the radius of the equivalent spherical charge R_{03}.

After substituting the numerical values, we obtain

$$0{,}65 = \frac{14\,700}{\left(\frac{100}{R_{03}}\right)^{1,13}} \frac{1}{250} \frac{2\frac{7{,}7\cdot10^3 R_{03}}{1500\cdot0{,}27}}{\frac{7{,}7\cdot10^3 R_{03}}{1500\cdot0{,}27} + 2{,}4} =$$

$$= \frac{118}{\left(\frac{100}{R_{03}}\right)^{1,13}} \frac{19 R_{03}}{19 R_{03} + 2{,}4}.$$

Solving this equation, we get $R_{03} = 1.1$ m.

The weight of the TNT charge will be

$$G = R_{03}/(0.053)^3 = (1.1\cdot10^6)/150 = 735\cdot10^3 \ kg = 7.35 \ m.$$

§13. APPROXIMATE ACCOUNT OF CAVITATIONAL PHENOMENA IN THE INTERACTION BETWEEN AN UNDERWATER SHOCK WAVE AND A PLATE

The very simple schemes of interaction between a shock wave and a partition, considered in the two preceding sections, did not take into account the cavitational phenomena in the liquid. Yet such phenomena can occur in many cases of practical interest.

Let us illustrate this by using as an example the motion of a free plate. We shall assume that on the one side of the plate there is water, and on the other there is air. The acoustic impedance of air can be neglected compared with the acoustic impedance of water.

Under such conditions, the differential equation of motion of the plate will be

$$m\frac{d^2z}{dt^2} + \rho_0 a_0 \frac{dz}{dt} = 2p_m e^{-\frac{t}{\tau}}. \tag{4.180}$$

Its solution is (see §11)

$$z = \frac{2 p_m \theta^2}{m \beta (\beta - 1)} \left[\beta - 1 + e^{-\frac{\beta t}{\theta}} - \beta e^{-\frac{t}{\theta}} \right]; \qquad (4.135)$$

$$\frac{dz}{dt} = \frac{2 p_m \theta}{m (\beta - 1)} \left[e^{-\frac{t}{\theta}} - e^{-\beta \frac{t}{\theta}} \right], \qquad (4.136)$$

where

$$\beta = \frac{\rho_0 a_0}{m} \theta. \qquad (4.181)$$

The maximum displacement of the plate does not depend on its inertial properties and is determined by the equation

$$z_{max} = \lim_{t \to \infty} z = \frac{2 p_m \theta}{\rho_0 a_0}. \qquad (4.182)$$

The resultant pressure on the plate consists of the pressure in the direct and reflected waves:

$$p_{pe3} = p_{np} + p_{otp} \qquad (4.183)$$

with

$$p_{otp} = p_{np} - \rho_0 a_0 \frac{dz}{dt} =$$

$$= p_m e^{-\frac{t}{\theta}} - \rho_0 a_0 \frac{2 p_m \theta}{m (\beta - 1)} e^{-\frac{t}{\theta}} + \rho_0 a_0 \frac{2 p_m \theta}{m (\beta - 1)} e^{-\beta \frac{t}{\theta}} =$$

$$= p_m \left\{ -\frac{\beta + 1}{\beta - 1} e^{-\frac{t}{\theta}} + \frac{2\beta}{\beta - 1} e^{-\beta \frac{t}{\theta}} \right\}. \qquad (4.184)$$

If we take into account the wave character of propagation of the disturbances and consider for the sake of simplicity a plane wave, then the pressure at an arbitrary point of the liquid ahead of the plate will be the result of the superposition of two wave systems, the direct and reflected waves.

Reckoning the time from the instant of arrival of the wave at the plate, we have

$$p_{pe3}(-x, t) = p_{np}(-x, t) \sigma_0 \left(t + \frac{x}{a_0} \right) + p_{otp}(-x, t) \sigma_0 \left(t - \frac{x}{a_0} \right) =$$

$$= p_m \left| e^{-\frac{\left(t + \frac{x}{a_0} \right)}{\theta}} \sigma_0 \left(t + \frac{x}{a_0} \right) - \left[\frac{\beta + 1}{\beta - 1} e^{-\frac{\left(t - \frac{x}{a_0} \right)}{\theta}} + \right. \right.$$

$$\left. \left. + \frac{2\beta}{\beta - 1} e^{-\beta \frac{t - \frac{x}{a_0}}{\theta}} \right] \sigma_0 \left(t - \frac{x}{a_0} \right) \right|. \qquad (4.185)$$

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The structure of the obtained equation enables us to plot the ratio p_{rez}/p_m in space-time coordinates for a fixed value of the parameter β. In the limiting case as $\beta \to 0$ we have an absolutely rigid wall, and when $\beta \to \infty$ we have the free surface of the liquid. Figure 105 shows the lines of equal pressures as functions of the coordinates and the time for $\beta = 4.0$ and $\beta = 83$.

It is easy to note that at some definite instant of time, even for plates of relatively large inertia ($\beta = 4.0$), the resultant values of the pressures in the liquid become negative.
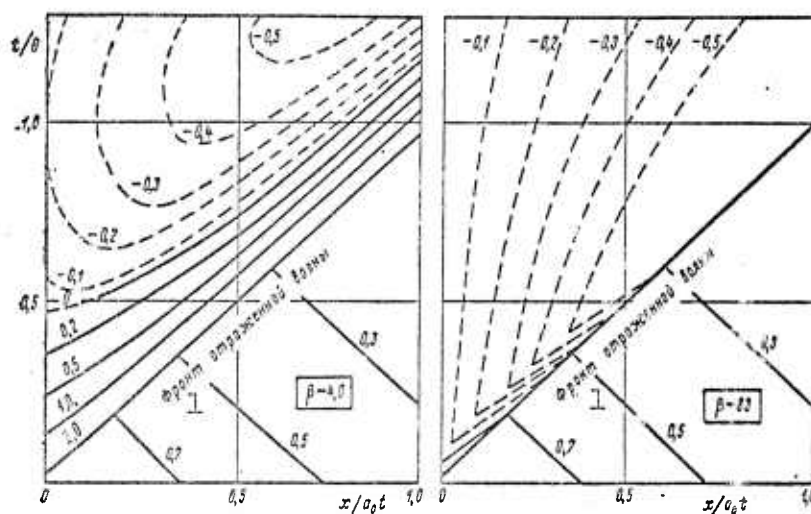

Fig. 105. Resultant pressure ahead of the plate in coordinates $\underline{x}$ and $\underline{t}$. 1) Front of reflected wave.

If the absolute values of the pressures turn out to be larger than the sum of the cavitation and hydrostatic pressures, then tensile stresses can arise in the liquid, capable of producing a cavitational discontinuity.

The formation and development of the cavitation under interactions between and underwater shock wave and a plate, were first considered by Schauer and Kirkwood. Schauer's theory is based on the assumption that starting with the instant when the resultant pressure (p_{rez}) is equal to the cavitation pressure

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the plate breaks away from the liquid and moves by inertia in the air medium.

This instant of time can be readily determined with the aid of the previously established relations.

Indeed, if we assume approximately, for example, that the cavitation sets in when the resultant pressure on the plate is equal to zero, then we get according to (4.183) and (4.185)

$$e^{-\frac{t_k}{\theta}} - \frac{\beta+1}{\beta-1} e^{-\frac{t_k}{\theta}} + \frac{2\beta}{\beta-1} e^{-\beta \frac{t_k}{\theta}} = 0,$$

hence

$$t_k = \theta \frac{\ln \beta}{\beta-1}, \tag{4.186}$$

where t_k is the instant of occurrence of cavitation.

The velocity of motion of the plate at the instant $t = t_k$ will, in accord with (4.136), be

$$\frac{dz}{dt}\bigg|_{t=t_k} = \frac{2 p_m \theta}{m(\beta-1)} \left[e^{-\frac{\ln \beta}{\beta-1}} - e^{-\beta \frac{\ln \beta}{\beta-1}} \right]. \tag{4.136a}$$

The kinetic energy of the free plate is absorbed by the work of the atmospheric counterpressure forces.

Schauer's theory as applied to problems involving the interaction between an underwater shock wave and ship structures was developed in the works of D.A. Aleksandrin.

Kirkwood, unlike Schauer, supposed that the maximum displacement of the plate is determined not only by the value of the kinetic energy which is imparted to it by the shock wave by the instant of formation of the cavitation region, but also by the energy of the cavitated layer of liquid.

Kirkwood's researches were continued by B.V. Zamyshlyayev. The scheme of development of the cavitational processes as the plate moves

under the action of the shock wave turns out to be quite close to that described previously in §4 of Chapter 3. The only difference is that in the study of the reflection of the wave from the free surface one could assume that the formation of the cavitation discontinuity coincides in time with the instant of arrival of the front of the reflected wave.

For an approximate estimate of the characteristics of the motion of the cavitational layer and of the plate, we make use, as before, of the laws of conservation of energy and momentum. Let us consider a system consisting of a plate and several cavitational layers of thickness h_i.

By the instant t_i when the i-th layer has separated, such a system acquires a momentum equal to the impulse of the pressure forces acting from the instant of arrival of the front of the shock wave at the plane $-h_i$ $(t = -h_i/a_0)$:

$$k_{i_0} = \int_{-\frac{h_i}{a_0}}^{t_i} p_{np}(t)\, dt + \int_{\frac{h_i}{a_0}}^{t_i} p_{orp}(t)\, dt = I_{ap}(i) + I_{orp}(i),^{*} \qquad (4.187)$$

where $I_{pr}(i)$ and $I_{otr}(i)$ are the impulses of the pressures in the direct and reflected waves, reckoned from the instant of arrival of these waves at the point h_i to the instant of formation of the cavitational discontinuity.

After the i-th layer is separated, the system under consideration is acted upon only by the forces of atmospheric pressure from the plate (the saturated-vapor plate from the i-th layer can be neglected).

For this reason, the momentum of the system at some instant of time $t > t_i$ will be

$$k_i(t) = k_{i_0}(t_i) - p_{атм}(t - t_i). \qquad (4.188)$$

Assuming that the cavitational layers and the plate move with

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some average velocity $z_1(t)$, we obtain

$$\dot{z}_1(t) = \frac{k_l(t)}{m + \rho_0 h_l} = \frac{I_{np}(t) + I_{otp}(t) - (t - t_l) p_{atm}}{m + \rho_0 h_l}. \qquad (4.189)$$

The kinetic energy of motion of the system will be

$$T_l(t) = \frac{m + \rho_0 h_l}{2} \dot{z}_1^2(t). \qquad (4.190)$$

It is obvious that this quantity should be equal to the energy of the direct and reflected waves, transferred to the system, after subtracting the work done by the forces of atmospheric pressure.*

The energy of the direct wave is determined by the equation

$$E_{np}(t) = \int_{-\frac{h_l}{a_0}}^{t_l} \frac{p_{np}^2}{\rho_0 a_0} dt. \qquad (4.191)$$

The energy of the reflected wave is

$$E_{otp}(t) = \int_{\frac{h_l}{a_0}}^{t_l} \frac{p_{otp}^2(t)}{\rho_0 a_0} dt. \qquad (4.192)$$

The work done by the forces of atmospheric pressure is approximately equal to

$$A = p_{atm}[z_1(t) - z_l(t_l)]. \qquad (4.193)$$

Gathering together the obtained estimates, we arrive at the relation

$$T_l(t) = E_{np}(t) - E_{otp}(t) - A, \qquad (4.194)$$

which is equivalent to

$$\frac{m + \rho_0 h_l}{2} \dot{z}_1^2(t) = E_{np}(t) - E_{otp}(t) - p_{atm}[z_1(t) - z_l(t_l)]. \qquad (4.195)$$

The value of $z_i(t_i)$ can be readily expressed in terms of the parameters of the direct and reflected waves.

Since

$$p_{otp} = p_{np} - \rho_0 a_0 \frac{dz}{dt},$$

$$\frac{dz}{dt} = \frac{p_{np} - p_{otp}}{\rho_0 a_0};$$

we have

$$z_i(t_i) = \int\limits_{-\frac{h_i}{a_s}}^{t_i} \frac{p_{np}}{\rho_0 a_0} dt - \int\limits_{\frac{h}{a_s}}^{t_i} \frac{p_{o\tau p}}{\rho_0 a_0} dt =$$

$$= \frac{I_{np}(l) - I_{o\tau p}(l)}{\rho_0 a_0}. \qquad (4.196)$$

Thus, to determine the displacement of the plate $z_1(t)$ we have

$$z_1(t) = \frac{I_{np}(l) - I_{o\tau p}(l)}{\rho_0 a_0} + \frac{E_{np}(l) - E_{o\tau p}(l)}{p_{a\tau M}} -$$

$$- \frac{m + \rho_0 h_l}{2p_{a\tau M}} \dot{z}_i^2(t). \qquad (4.197)$$

Let us calculate the maximum displacement of the plate z_{max}.

To this end it is necessary to determine the number of cavitational layers N which have had time to attach themselves to the plate by that time, and equate the velocity $\dot{z}_1(t)$ to zero.

According to (4.189), the instant of time at which the maximum displacement is reached, is equal to

$$t_m = t_N + \frac{I_{np}(N) + I_{o\tau p}(N)}{p_{a\tau M}}, \qquad (4.198)$$

with the N-th layer becoming attached to the system when its displacement turns out to be equal to the displacement of the plate (with the N — 1 layers adjacent to it).

The displacement of the plate is determined by Eq. (4.197) in which the index i is replaced by the index N.

The displacement of the N-th layer can be calculated from the relation

$$z_N(t) = z_N(t_N) + \dot{z}_{N0} \cdot (t - t_N) =$$

$$= \frac{I_{np}(N) - I_{o\tau p}(N)}{\rho_0 a_0} + \frac{2p_{np}(N) + |p_k| + p_0}{\rho_0 a_0} (t - t_N). * \qquad (4.199)$$

Equating the velocities, we arrive at a quadratic equation in the unknown instant of time. Solving this equation, we obtain

$$t_N^. = t_N + b_N + \sqrt{b_N^2 + C_N}, \qquad (4.200)$$

where

$$b_N = \frac{I_{np}(N) + I_{отр}(N)}{p_{атм}} -$$

$$- \frac{m + \rho_0 h_N}{\rho_0 a_0 p_{атм}} [2p_{np}(N) + |p_к| + p_0]; \qquad (4.201)$$

$$c_N = \frac{2(m + \rho_0 h_N)}{p_{атм}^2} [E_{np}(N) - E_{отр}(N)] -$$

$$- \frac{[I_{np}(N) + I_{отр}(N)]^2}{p_{атм}^2}. \qquad (4.202)$$

An estimate of the cavitational layers can be carried out by comparing t^*_N and t_m.

Equating these instants of time, we obtain the following transcendental equation

$$E_{np}(N) - E_{отр}(N) = \frac{1}{\rho_0 a_0} [2p_{np}(N) + |p_к| + p_0] \times$$

$$\times [I_{np}(N) + I_{отр}(N)]. \qquad (4.203)$$

In addition, we previously had

$$p_{рез} = p_{np}(N) + p_{отр}(N) = -(|p_к| + p_0). \qquad (4.204)$$

These two equations include two unknown quantities h_N and t_N, and once these are determined it is easy to calculate the momenta and the energies in the direct and reflected waves, determining by the same token all the elements of motion.

According to (4.197), the maximum displacement of the plate will be

$$z_{max} = \frac{I_{np}(N) - I_{отр}(N)}{\rho_0 a_0} + \frac{E_{np}(N) - E_{отр}(N)}{p_{атм}}. \qquad (4.205)$$

The results obtained can be extended to the case of a plate which is part of the whole structure. In this case it is necessary to estimate in addition the value of the potential deformation energy, which is determined with the aid of the known methods of structural mechanics.

Without dwelling on the details of this problem, we note in conclusion that an account of cavitational phenomena leads to an increase

in the maximum deflection of the plate as compared with design in which cavitation is not taken into account.

Manu-script Page No.	[Footnotes]
251	In the case of great excess pressures at the front of the air shock wave, supersonic gas flow occurs and a compression wave is formed about the partition. The concept of stagnant flow parameters takes on new significance.
258	As is well known, the hydraulic friction factor can be used to express the resistance of the tube in terms of the impact pressure: $\Delta p = \zeta \frac{l}{d} \frac{\rho v^2}{2}.$ (4.16)
263	For further details on the application of the lagrange method to problems of dynamic structural strength, see §9.
273	The result discussed below is taken from K. V. Lopukhov.
274	We will leave out the structure of the solution, since it is completely analogous to the solution presented in the book by A.A. Kharkevich, "Nonsteady-state Wave Phenomena."
281	This problem was examined by us jointly with O.G. Fayans.
297	Here we neglect the defraction of the shock wave from the plate edges.
297	The terminology "vertical" and "horizontal" partitions that we employed is somewhat conditional, in more rigorous formulations, we speak of the case in which the shock wave drops along the normal to the surface or slides along the surface.
308	The coefficient $\bar{\alpha}$ has the same sign as k_{22}; $$\alpha > 0, \text{ if } \rho_3 a_3 > \rho_2 a_2;$$ $$\alpha < 0, \text{ if } \rho_3 a_3 < \rho_2 a_2;$$ the coefficient γ has the sign of the product $(\rho_3 a_3 - \rho_2 a_2) \times (\overline{\rho}_1 a_1 - \rho_2 a_2).$
327	It is appropriate to emphasize that the time of formation of the i-th cavitational layer does not coincide, generally speaking, with the instant of arrival of the reflected wave at the plane h_i.

328   It must be borne in mind that whereas the direct wave brings
some energy into the system, the reflected wave, to the con-
trary, carries energy away. Consequently, the system will
have an energy $E_{pr}(i) - E_{otr}(i)$.

329    $$* \; p_{рез} = -(|p_к| + p_0) = 2p_{пр} - \rho_0 a_0 \frac{dz}{dt}\,.$$

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